CONVEX BODIES AND ALGEBRAIC EQUATIONS ON AFFINE VARIETIES

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Contents

Key words: Affine variety, system of algebraic equations, Hilbert function, convex body, degree of a line bundle, Alexandrov-Fenchel inequality, Brunn-Minkowski inequality. Newton polytope.

Note: This is a preliminary version and may contain several typos.

1. Introduction

The theory of Newton polytopes relates algebraic geometry of subvarieties in $(\mathbb{C}^*)^n$ and convex geometry (for a survey see for example [Khov2]). In other words, this is a connection between the theory of toric varieties and geometry of convex polytopes. In this paper we discuss a much more general connection between algebraic geometry and convex geometry. This connection is useful in both directions. It yields new, simple and transparent proofs of a series of classical results (which are not considered as simple) both in algebraic geometry and in convex geometry.

We prove the following classical results from algebraic geometry: Hodge Index Theorem (according to which the square of intersection index of two algebraic curves on an irreducible algebraic surface is greater than or equal to the product of self intersections of the curves), Kushnirenko–Bernstein theorem on the number of roots of generic system of algebraic equations with fixed Newton polyhedra. We also develop a version of intersection theory for (quasi) affine varieties. We show that properties of number of solutions of a generic system of equations on an n-dimensional (quasi) affine algebraic variety resemble, in many ways, the properties of mixed volumes of n convex bodies in $\mathbb{R}^n$. In the part related to convex geometry we prove Alexander–Fenchel inequality — which is one of the main inequalities concerning mixed volumes. Many other geometric inequalities follow as its corollaries.

In our proofs we use simple and rather restricted tools. From algebraic geometry we use classical Hilbert theory on degree of subvarieties of projective space (see Section 4.2 for statement of Hilbert theorem, its proof could be found in most of the textbooks in algebraic geometry for example [Harris, Lecture 13]). From convex geometry we use Brunn-Minkowski inequality. It is actually enough for us to use the classical isoperimetric inequality which is Brunn-Minkowski for convex domains in plane.

About the content of the paper. In Sections 1-7 we construct a version of intersection theory for (quasi) affine varieties. To a (quasi) affine variety

X we associate a set $K(X)$. By definition each element in $K(X)$ is a finite dimensional space L of regular functions on X, such that for any point in X at least one function from L is not equal to zero. Product L_1L_2 of two spaces $L_1, L_2 \in K(X)$ is the space spanned by the functions f_1f_2, where $f_1 \in L_1$, $f_2 \in L_2$. The set $K(X)$ equipped with this multiplication become a commutative semigroup. In Section 2.3 we introduce an intersection index in the semigroup $K(X)$, where now X is an irreducible (quasi) affine n-dimensional variety. The intersection index of an n-tuple $L_1, \ldots, L_n \in K(X)$, denoted by $[L_1, \ldots, L_n]$, is the number of solutions of a sufficiently general system of equations $f_1 = \cdots = f_n = 0$ on X, where $f_1 \in L_1, \ldots, f_n \in L_n$.

We show that for almost all n-tuples $f_1 \in L_1, \ldots, f_n \in L_n$, number of solutions of the system $f_1 = \cdots = f_n = 0$ is the same and hence the intersection index is well-defined. In Section 2.2 we state some classical results which we need for the proof of this fact. The properties of the intersection index are similar to the properties of mixed volumes of n convex bodies. Some of these properties could be deduced from the case in which X is an algebraic curve. (see Sections 4-6). But to prove the most interesting property, namely an analogue of Alexandrov–Fenchel inequality, we have to consider algebraic surfaces (see Section 2.7). The corresponding property of algebraic surfaces is proved Section 5.3.

In Section 4.3 we associate a convex body to each space $L \in K(X)$ where X is an irreducible n-dimensional (quasi) affine variety. We will show that, under some small extra assumptions, the volume of this convex body multiplied by $n!$ is equal to the self intersection index $[L, \ldots, L]$ of the space L. This construction provides the relation between algebraic geometry and convex geometry in this paper. Let us describe this construction more precisely.

First we fix a $\mathbb{Z}^n$-valued valuation on the field of rational functions on X. There are many different valuations of this kind (see Section 4.1). Different valuations associate different convex bodies to a space $L \in K(X)$. The body is constructed as follows: for each $k \in \mathbb{N}$, values of the valuation on the space L^k belong to a finite subset $\tilde{G}_k(L)$ in the group $\mathbb{Z}^n$. The number of points in the set $\tilde{G}_k(L)$ is equal to the dimension of the space L^k. Let us add a first coordinate equal to k to all points in $\tilde{G}_k(L)$. We obtain a new set $G_k(L) \subset \mathbb{Z} \times \mathbb{Z}^n$. The union over k of all the sets $G_k(L)$ is a semigroup $G(L)$ in $\mathbb{Z} \times \mathbb{Z}^n$. Let us consider the smallest convex cone C (centered at origin) in $\mathbb{R} \times \mathbb{R}^n \supset \mathbb{Z} \times \mathbb{Z}^n$, which contains the semigroup $G(L)$. The intersection of the cone C with the hyperplane $k = 1$ is our desired convex body $\Delta(G(L))$ associated with the space L. We will show that the set of all integral point in the cone C provides a very good approximation of the semigroup $G(L)$. After that the relation between the volume of the body $\Delta(G(L))$ and the self intersection $[L, \ldots, L]$ follows from the Hilbert theorem (see Section 4.2).

The usual Newton polytope associated to a Laurent polynomial is a very special case of this construction (see Section 5.1). The Newton polytopes are

naturally related to toric varieties. Interestingly, the Gelfand-Cetlin polytopes of irreducible representations of $\mathrm{GL}(n,\mathbb{C})$, and more generally string polytopes for irreducible representations of a connected reductive group G, also appear as the Newton convex body $\Delta(G(L))$. As X we take the flag variety, or rather the open affine Schubert cell in it (Example 4.23). For this, see [Ok2] for Gelfand-Cetlin polytopes of $G = \mathrm{SP}(2n,\mathbb{C})$ and [Kaveh] for the general case.

The results we need on semigroups of integral points are proved in Sections 3.3-3.4. In Section 3.3 we not only estimate the number of integral points with fixed first coordinate but also estimate the sum of value of a polynomial over this subset in the semigroup $G(L)$. We won't need this estimation of the sum of values of a polynomial in this paper, although it will be used in the next paper [K-Kh]. In [K-Kh] we consider a variety X, equipped with a reductive group action and a subspace L of regular functions on X invariant under this action. We will prove a generalization of Kazarnovskii-Brion formula (for the degree of a normal projective spherical variety) to (quasi) affine, not necessarily normal, spherical varieties.

The results of Section 3.3-3.4 use the facts from convex geometry which we prove in Section 3.2.

In Section 5.1 we briefly show that the well-known Kushnirenko and Bernstein theorems follow from our general results. In fact the proof in Section 5.1 almost coincides with the proof in [Khov1]. Bernstein theorem relates mixed volume with number of solutions of a generic system of Laurent polynomial equations. In Section 5.3 using the isoperimetric inequality (Brunn–Minkowski inequality for planar convex bodies) we prove an algebraic analogous of Alexandrov–Fenchel inequality and its numerous corollaries. In Section 5.4 we show that the the corresponding geometric inequalities follows from their algebraic analogues.

If X is an affine algebraic curve one can describe the geometry of the semigroup $G(L)$, $L \in K(X)$ in detail (see Section 5.5).

We should point out that the assumption that X is (quasi) affine is not crucial for the results of this paper. In fact, one can take X to be any irreducible variety and replace L with a subspace of section of a line bundle on X. Given a valuation on the ring of sections of the line bundle, in the same way one constructs a convex body associated to (X, L). The same arguments as in the paper then can be used to give a relation between the volume of this convex body and the self-intersection number of a generic section from L.

This paper is our first work in a series of papers under preparation, dedicated to the new relation between convex geometry and algebraic geometry.

2. An intersection theory for affine varieties

2.1. **Semigroup of subspaces of a ring of functions on a set.** We start with some general definitions. *A set equipped with a ring of functions* is a set

X, with a ring $R(X)$ consisting of complex valued functions, containing all complex constants. To a pair $(X, R(X))$ one can associate the set $VR(X)$ whose elements are vector subspaces in $R(X)$.

There is a natural multiplication in $VR(X)$. For any two subspaces $L_1, L_2 \subset R(X)$ define *product* L_1L_2 to be the linear span of functions fg, where $f \in L_1$ and $g \in L_2$. With this product the set $VR(X)$ becomes a commutative semigroup.

Let us say that a subspace L *has no common zeros on* X, if for each $x \in X$ there is a function $f \in L$ with $f(x) \neq 0$.

Proposition 2.1. *Let L_1, L_2 be vector subspaces in $R(X)$. If L_1, L_2 have finite dimension (respectively, if each subspace L_1, L_2 has no common zeros on X), then the space L_1L_2 is finite dimensional (respectively, the space L_1L_2 has no common zeros on X).*

Proof. 1) Let $\{f_i\}$, $\{g_j\}$ be bases for subspaces L_1,L_2. Then the functions $\{f_ig_j\}$ span the space L_1L_2. So, if L_1, L_2 have finite dimension then, L_1L_2 also has a finite dimension. 2) If the functions $f_1 \in L_1$, $f_2 \in L_2$ do not vanish at a point $x \in X$, then the function $f_1f_2 \in L_1L_2$ does not vanish at x and thus if each space L_1, L_2 has no common zeros on X, then the space L_1L_2 also has no common zeros on X. $\square$

According to Proposition 2.1, *subspaces of finite dimension in $R(X)$, each of which has no common zeros on X form a semigroup in $VR(X)$ which we will denote by $KR(X)$.*

Assume that $Y \subset X$ and that the restriction of each function $f \in R(X)$ to the set Y belongs to a ring $R(Y)$. We will denote the restriction of a subspace $L \subset R(X)$ to Y by the same symbol L. Clearly if $L \in KR(X)$ then $L \in KR(Y)$.

In this paper we will not use general sets equipped by rings of functions. Instead the following example plays a main role.

Example 2.2. Let X be a complex (quasi) affine algebraic variety and let $R(X)$ be the ring of regular functions on X. In this case to make notations shorter we will not mention the ring $R(X)$ explicitly and the semigroup $KR(X)$ will be denoted by $K(X)$.

Any subspace $L \in K(X)$ gives a natural map $\Phi_L : X \to \mathbb{P}(L^*)$, where L^* denotes the vector space dual of L. For $x \in X$ define $\xi \in L^*$ by

$$\xi(f) = f(x),$$

for all $f \in L$. Since the elements of L have no common zero, $\xi \neq 0$. Let $\Phi_L(x)$ be the point in $\mathbb{P}(L^*)$ represented by ξ. Fix a basis $\{f_1, \dots f_d\}$ for L. One verifies that the map Φ_L in the homogeneous coordinates in $\mathbb{P}(L^*)$, corresponding to the dual basis to the f_i, is given by

$$\Phi_L(x) = (f_1(x) : \dots : f_d(x)).$$

A subspace $L \in K(X)$ is called *very ample* if Φ_L is an embedding.

Finally let us say that a regular function $f \in R(X)$ *satisfies an integral algebraic equation over a space* $L \in K(X)$, if

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0,$$

where m is a natural number and $a_i \in L^i$, for each $i = 1, \ldots, m$.

2.2. **Preliminaries on affine algebraic varieties.** Now we discuss some facts needed to define an intersection index in the semigroup $K(X)$. We will need particular cases of the following results: 1) An affine algebraic variety has a *finite* topology; 2) There are finitely many topologically different varieties in an algebraic family of affine varieties; 3) In such a family the set of parameters, for which the corresponding members have the same topology, is a complex semi-algebraic subset in the space of parameters; 4) A complex semi-algebraic subset in a vector space covers almost all of the space, or covers only a very small part of it.

Now let us give exact statements of these results and their particular cases we will use.

Let X, Y be complex affine algebraic varieties and let $\pi : X \to Y$ be a regular map. Consider a family of affine algebraic varieties $X_y = \pi^{-1}(y)$, parameterized by points $y \in Y$. The following theorem is well-known.

Theorem 2.3. *Each variety X_y has a homotopy type of a finite CW-complex. There is a finite stratification of the variety Y into complex semi-algebraic strata Y_α, such , that for points y_1, y_2 belonging to the same stratum Y_α varieties X_{y_1}, X_{y_2}, are homeomorphic (In particular, in the family X_y there are one finitely many topologically different varieties.)*

When X, Y are real affine algebraic varieties and $\pi : X \to Y$ is a regular real map, a similar statement holds. One can also extend it to some other cases of varieties and maps (see [Dries]). We will need only the following simple corollary of this theorem for which we give sketch of a proof (independent of the above theorem).

Let $L_1, \ldots, L_n$ be finite dimensional subspaces in the space of regular functions on an n-dimensional complex affine algebraic variety X. Denote by $X_{\mathbf{f}}$, where $\mathbf{f} = (f_1, \ldots, f_n)$ is a point in $\mathbf{L} = L_1 \times \cdots \times L_n$, the subvariety of X, defined by the system of equations $f_1 = \cdots = f_n = 0$. In the space $\mathbf{L}$ of parameters consider the subset $\mathbf{F}$ consisting of all parameters $\mathbf{f}$ such that the set $X_{\mathbf{f}}$ contains isolated points only.

Corollary 2.4. *1) If $\mathbf{f} \in \mathbf{F}$, then the set $X_{\mathbf{f}}$ contains finitely many points. Denote the number of points in $X_{\mathbf{f}}$ by $k(\mathbf{f})$; 2) Function $k(\mathbf{f})$ on the set $\mathbf{F}$ is bounded; 3) The subset $\mathbf{F}_{\max} \subset \mathbf{F}$ on which the function $k(\mathbf{f})$ attains its maxima is a complex semi-algebraic subset in $\mathbf{L}$.*

Sketch of proof (independent of above theorem). One can assume, that the variety X is defined in a space $\mathbb{C}^N$ by a non degenerated system of polynomial equations $g_1 = \cdots = g_{N-n} = 0$. (To be exact, X can be covered by a finite collection of Zariski open domains and in each domain X is defined in

such a way, see [Wh]. It is then enough to estimate the number of roots in each domain). One can also assume that all the functions belonging to the spaces $L_1, \dots, L_n$ are restrictions of polynomials on $\mathbb{C}^N$ to X belonging to a finite dimensional space $\bar{L}$. Let M be the maximum degree of all the polynomial g_i and the all polynomials in $\bar{L}$. From the classical Bezout theorem it is easy to deduce that $k(\mathbf{f}) < M^N$. Using the complex version of Tarski theorem one can prove that the function $k(\mathbf{f})$ takes finitely many values c and each level set $\mathbf{F}_c$ is semi-algebraic. (Similar fact is true in real algebraic geometry. One proves it using Tarski theorem. For an elementary proof of Tarski theorem see [B-Kh]). □

We will need the following simple property of complex semi-algebraic sets.

Proposition 2.5. *Let $F \subset L$ be a complex semi-algebraic subset in a vector space L. Then either there is an algebraic hypersurface $\Sigma \subset L$ which contains F, or F contains a Zariski open set $U \subset L$.*

We will use this proposition in the following form.

Corollary 2.6. *Let $F \subset L$ be a complex semi-algebraic subset in a vector space L. In the following cases F contains a (non-empty) Zariski open subset $U \subset L$: 1) F is an everywhere dense subset of L, 2) F does not have zero measure.*

2.3. **An intersection index in semi-group $K(X)$.**

Definition 2.7. Let X be a complex n-dimensional (quasi) affine algebraic variety and let $L_1, \dots, L_n$ be elements of $K(X)$. The *intersection index* $[L_1, \dots, L_n]$ of $L_1, \dots, L_n \in K(X)$ is the maximum of number of roots of a system $f_1 = \cdots = f_n$ over all the points $\mathbf{f} = (f_1, \dots, f_n) \in L_1 \times \cdots \times L_n = \mathbf{L}$, for which corresponding system has finitely many solutions.

By Corollary 2.4 the maximum is attained and the previous definition is well-defined.

Theorem 2.8 (Obvious properties of the intersection index)**.** *(1) $[L_1, \dots, L_n]$ is a symmetric function of the n-tuples $L_1, \dots, L_n$ (i.e. takes the same value under a permutation of the elements $L_1, \dots, L_n$), (2) it is monotone, (i.e. if $L'_1 \subset L_1, \dots, L'_n \subset L_n$, then $[L_1, \dots, L_n] \geq [L'_1, \dots, L'_n]$) and (3) non-negative (i.e. $[L_1, \dots, L_n] \geq 0$).*

Theorem 2.8 is a straight forward corollary from the definition.

Let X be a complex n-dimensional (quasi) affine algebraic variety, let $k \in \mathbb{N}$ and let $L_1, \dots, L_k$ be an n-tuple of subspaces belonging to the semigroup $K(X)$. Put $\mathbf{L} = L_1 \times \cdots \times L_k$.

Proposition 2.9. *There is a Zariski open domain $\mathbf{U}$ in $\mathbf{L}$ such that for each point $\mathbf{f} = (f_1, \dots, f_k)$ in $\mathbf{U}$ the system of equations $f_1 = \cdots = f_k = 0$ on X is non degenerate (that is, at each root of the system the covectors $df_1, \dots, df_k$ are linearly independent).*

Proof. Fix a basis $\{g_{i,j}\}$ for each space L_i. Consider all the k-tuples $\mathbf{g_j} = (g_{1,j_1}, \dots, g_{k,j_k})$, where $\mathbf{j} = (j_1, \dots, j_k)$, containing exactly one vector from each of the bases for the L_i. Denote by $V_\mathbf{j}$ the Zariski open domain in X defined by the system of inequalities $g_{1,j_1} \neq 0, \dots, g_{k,j_k} \neq 0$. The union of the sets $V_\mathbf{j}$ coincides with X, because $L_1, \dots, L_k \in K(X)$. In the domain $V_\mathbf{j}$ rewrite the system $f_1 = \dots = f_k = 0$ as follows: represent each function f_i in the form $\bar{f_i} = \bar{f_i} + c_i g_{i,j_i}$, where $\bar{f_i}$ belongs to the linear span of all the vectors $g_{i,j}$ excluding the vector g_{i,j_i}. Now in $V_\mathbf{j}$ the system could be rewritten as $\frac{\bar{f_1}}{g_{1,j_1}} = -c_1, \dots, \frac{\bar{f_k}}{g_{k,j_k}} = -c_k$. According to Sard's theorem, for almost all the $\mathbf{c} = (c_1, \dots, c_k)$ the system is non-degenerate. Denote by $W_\mathbf{j}$ the subset in $\mathbf{L}$, consisting of all $\mathbf{f}$ such that the system $f_1 = \dots = f_k = 0$ is non-degenerate in $V_\mathbf{j}$. We have proved that the set W_j is a set of full measure in $\mathbf{L}$. On the other hand the set W_j is a complex semi-algebraic subset in $\mathbf{L}$. Thus, according to Corollary 2.6, W_j contains a Zariski open subset $\mathbf{U_j}$. The intersection $\mathbf{U}$ of the sets $\mathbf{U_j}$ is a Zariski open subset in $\mathbf{L}$ which satisfies all the requirements of Proposition 2.9. □

Proposition 2.10. *The number of isolated roots of a system $f_1 = \dots = f_n = 0$, where $f_1 \in L_1, \dots, f_n \in L_n$, counted with multiplicity, is smaller than or equal to $[L_1, \dots, L_n]$.*

Proof. Let A be the set of isolated roots of our system. Let $k(A)$ be the sum of multiplicities of roots in A. According to Proposition 2.9 one can perturb the system a little bit to make it non-degenerate. Under such a perturbation the roots belonging to the set A will split into $k(A) > [L_1, \dots, L_n]$ simple roots. Thus we get a non-degenerate system with finitely many simple roots. By Corollary 2.4 the number of these roots can not be bigger than $[L_1, \dots, L_n]$. □

Now we prove that if a system of equations is generic then instead of inequality in Proposition 2.10 we have an equality. As before let $\mathbf{L} = L_1 \times \dots \times L_n$.

Proposition 2.11. *There is a Zariski open domain $\mathbf{U}$ in $\mathbf{L}$ such that for each point $\mathbf{f} = (f_1, \dots, f_n)$ in $\mathbf{U}$ the system of equations $f_1 = \dots = f_n = 0$ on X is non degenerate and has exactly $[L_1, \dots, L_n]$ solutions.*

Proof. Proof First, if a system has $[L_1, \dots, L_n]$ many isolated roots than the system is non-degenerate, otherwise its number of roots counting with multiplicity is bigger than $[L_1, \dots, L_n]$, which is impossible by Proposition 2.10. So there must be a non-degenerate system which has $[L_1, \dots, L_n]$ solutions. Second, any sufficiently general system has exactly the same number of isolated roots and almost all sufficiently general systems are non degenerate. So the set of non degenerate systems which have exactly $[L_1, \dots, L_n]$ could not be a set of measure zero. But this set is complex semi-algebraic, so according to the corollary 2.2 it contains a Zariski open domain. □

For each k-dimensional (quasi) affine subvariety Y in X and for each k-tuple of spaces $L_1, \dots, L_k \in K(X)$ let $[L_1, \dots, L_k]_Y$ be the intersection index of the restrictions of these subspaces to Y.

Consider an n-tuple $L_1, \dots, L_n \in K(X)$. For $k \leq n$ put $\mathbf{L}(k) = L_1 \times \cdots \times L_k$. According to Proposition 2.9 there is a Zariski open subset $\mathbf{U}(k)$ in $\mathbf{L}(k)$ such that if $\mathbf{f}(k) = (f_1, \dots, f_k) \in \mathbf{U}(k)$ then the system $f_1 = \cdots = f_k = 0$ is non-degenerate and hence defines a smooth subvariety $X_{\mathbf{f}(k)}$ in X.

Theorem 2.12. *1) For each point $\mathbf{f}(k) \in \mathbf{U}(k)$ the following inequality holds*

$$[L_1, \dots L_n]_X \leq [L_{k+1}, \dots L_n]_{X_{\mathbf{f}(k)}}. \tag{1}$$

2) There is a Zariski open subset $\mathbf{V}(k) \subset \mathbf{U}(k)$, such that for each point $\mathbf{f}(k) \in V(k)$ the inequality (1) *in fact is an equality.*

Proof. 1) If for a point $\mathbf{f}(k)$ inequality (1) does not hold, then there are $f_{k+1} \in L_{k+1}, \dots, f_n \in L_n$ such that the system $f_1 = \cdots = f_k = f_{k+1} = \cdots = f_n = 0$ has more isolated solution on X than the intersection index $[L_1, \dots, L_n]$, which is impossible. 2) According to Proposition 2.9 the collection of systems $\mathbf{f} = (f_1, \dots, f_n) \in \mathbf{L}$ for which the subsystem $f_1 = \cdots = f_k = 0$ is non-degenerate contains a Zariski open domain $\mathbf{V} \subset \mathbf{L}$. Let $\pi : \mathbf{L} \to \mathbf{L}(k)$ be the projection $(f_1, \dots, f_n) \mapsto (f_1, \dots, f_k)$. Now we can take $\mathbf{V}(k)$ to be any Zariski open domain in $\mathbf{L}(k)$ contained in $\pi(\mathbf{V})$. $\square$

Theorem 2.12 allows us to reduce the computation of the intersection index on a high dimensional (quasi) affine variety to computation of the intersection index on a lower dimensional (quasi) affine subvariety. It is not hard to establish main properties of the intersection index for affine curves. Using Theorem 2.12 we will then obtain corresponding properties for the intersection index on (quasi) affine varieties of arbitrary dimension for free.

2.4. **Preliminaries on affine algebraic curves.** Here we present some basic facts about affine algebraic curves which we will use later. Let X be a smooth complex affine algebraic curve (not necessarily irreducible).

Theorem 2.13 (normalization of algebraic curves)**.** *There is a unique (up to isomorphism) smooth projective curve $\bar{X}$ which contains X. The complement $A = \bar{X} \setminus X$, is a finite set, and any regular function on X has a meromorphic extension to $\bar{X}$.*

One can find a proof of this classical result in most of the text books in algebraic geometry (e.g. [Hart, Chapter 1]). This theorem allows us to find the number of zeros of a regular function g on X which has a prescribed behavior at infinity i.e. $\bar{X} \setminus X$. Indeed if g is not identically zero on some irreducible component of the curve X, then the order $ord_a g$ of its meromorphic extension at a point $a \in \bar{X}$ is well-defined defined. Function g on the projective curve $\bar{X}$ has the same number of roots (counting with multiplicities) as the number of poles (counting with multiplicities). Thus we have the following.

Proposition 2.14. *For every regular function g on an affine algebraic curve X (which is not identically zero at any irreducible component of X) the number of roots counting with multiplicity is equal to $-\sum_{a\in A} ord_a g$, where $ord_a g$ is the order at the point a of the meromorphic extension of the function g to $\bar{X}$.*

2.5. **Intersection index in semigroup $K(X)$ of an affine algebraic variety X.** Let $L \in K(X)$ and let $B = \{f_i\}$ be a basis for L such that none of the f_i are identically equal to zero at any component of the curve X. For each point $a \in A = \bar{X} \setminus X$ denote by $ord_a L$ the minimum, over all functions in B, of the numbers $ord_a f_i$. Clearly for every $g \in L$ we have $ord_a g \geq ord_a L$. The collections of functions $g \in L$ whose order at the point a is strictly bigger than $ord_a L$ form a proper subspace L_a of L.

Definition 2.15. By definition *degree* of a subspace $L \in K(X)$ is $\sum\limits_{a\in A} -ord_a L$, and denoted by $\deg(L)$.

For each component X_j of the curve X denote the subspace in L, consisting of all the functions identically zero on X_i by L_{X_i}. The space L_{X_i} is a proper subspace in L because $L \in K(X)$.

The following is a corollary of Proposition 2.14.

Proposition 2.16. *If function $f \in L$ does not belong to the union of the subspaces L_{X_i}, then f has finitely many roots on X. The number of the roots of the function f, counted with multiplicity, is less than or equal to* $\deg(L)$. *If function f is not in the union of the subspaces L_a, $a \in A$, then the equality holds.*

Proposition 2.17. *For any two $L, G \in K(X)$ the following identity holds* $[L] + [G] = [LG]$.

Proof. For each point $a \in A$ and any two functions $f \in L$, $g \in G$ the identity $ord_a f + ord_a g = ord_a fg$ holds. As a corollary we have $ord_a L + ord_a G = ord_a LG$. So, $\deg(L) + \deg(G) = \deg(LG)$ and hence $[L] + [G] = [LG]$. □

Consider the map $-Ord$ which associate to a subspace $L \in K(X)$ an integral valued function on the set A, namely value of $-Ord(L)$ at $a \in A$ is equals $-ord_a L$. The map $-Ord$ is a homomorphism from the multiplicative semigroup $K(X)$ to the additive group of integral valued functions on the set A. Clearly the number $[L]$ can be computed in terms of the homomorphism $-Ord$ because $[L] = \deg(L) = \sum_{a\in A} -ord_a L$.

Proposition 2.18. *Assume that a regular function g on the curve X satisfies an integral algebraic equation over a subspace $L \in K(X)$. Then at each point $a \in A$ we have*

$$ord_a g \geq ord_a L.$$

Proof. Let $g^n + f_1 g^{n-1} + \cdots + f_n = 0$ where $f_i \in L^i$. Suppose $ord_a g = k < ord_a L$. Since $g^n = -f_1 g^{n-1} - \cdots - f_n$ we have $nk = ord_a g^n \geq$

$\min\{ord_a f_1 g^{n-1}, \dots, f_n\}$. That is, for some i, $nk \geq ord_a f_i + k(n-i)$. But for every i, $ord_a f_i g^{n-i} = ord_a f_i + ord_a g^{n-i} > i \cdot ord_a L + k(n-i) > nk$. The contradiction proves the claim. □

Corollary 2.19. *Assume that a regular function g on the curve X satisfies an integral algebraic equation over a subspace $L \in K(X)$. Consider the subspace $G \in K(X)$ spanned by g and L. Then: 1) At each point $a \in A$ the equality $ord_a L = ord_a G$ holds; 2) $[L] = [G]$; 3) For each subspace $M \in K(X)$ we have $[LM] = [GM]$.*

2.6. **Properties of the intersection index which can be deduced from the curve case.**

Theorem 2.20 (Multi-linearity)**.** *Let $L_1', L_1'', L_2, \dots, L_n \in K(X)$ and put $L_1 = L_1' L_1''$. Then*

$$[L_1, \dots, L_n] = [L_1'', \dots, L_n] + [L_1', \dots, L_n].$$

Proof. Consider three n-tuples $(L_1', \dots, L_n)$, $(L_1'', \dots, L_n)$ and $(L_1' L_1'', \dots, L_n)$ of elements of the semigroup $K(X)$. According to the Theorem 2.12 there is an $(n-1)$-tuple $f_2 \in L_2, \dots, f_n \in L_n$, such that the system $f_2 = \dots + f_n = 0$ is non-degenerate and defines a curve $Y \subset X$ such that $[L_1', \dots, L_n] = [L_1']_Y$, $[L_1'', \dots, L_n] = [L_1'']_Y$ and $[L_1' L_1'', \dots, L_n] = [L_1' L_1'']_Y$. Using Proposition 2.17 we now obtain $[L_1' L_1'']_Y = [L_1']_Y + [L_1'']_Y$ and theorem is proved. □

Theorem 2.21 (Integral closure property)**.** *Let $L_1 \in K(X)$ and let $G_1 \in K(X)$ be a subspace spanned by $L_1 \in K(X)$ and some regular functions g satisfying an integral algebraic equation over L_1. Then for any $(n-1)$-tuple $L_2, \dots, L_n \in K(X)$ we have*

$$[L_1, L_2, \dots, L_n] = [G_1, L_2, \dots, L_n].$$

Proof. Consider two n-tuples $(L_1, L_2, \dots, L_n)$ $(G_1, L_2, \dots, L_n)$ of $K(X)$. According to Theorem 2.12 there is a $(n-1)$-tuple $(f_2, \dots, f_n)$, $f_i \in L_i$, such that the system $f_2 = \dots + f_n = 0$ is non degenerate and defines a curve $Y \subset X$ such that $[L_1, L_2, \dots, L_n] = [L_1]_Y$, $[G_1, L_2 \dots, L_n] = [G_1]_Y$. Using Corollary 2.19 we obtain $[L_1]_Y = [G_1]_Y$ as required. □

2.7. **Properties of the intersection index which can deduced from the surface case.** Let $Y \subset X$ be a (quasi) affine subvariety and let $L \in K(X)$ be a very ample subspace. Then the restriction of functions from L to Y is a very ample space in $K(Y)$.

Theorem 2.22 (A version of Lefschetz theorem)**.** *Let X be a smooth irreducible n-dimensional (quasi) affine variety and let $L_1, \dots, L_k \in K(X)$, $k < n$, be very ample subspaces, i.e. the maps $\Phi_{L_i} : X \to PL_i$ are embeddings. Then there is a Zariski open set $\mathbf{U}(k)$ in $\mathbf{L}(k) = L_1 \times \dots \times L_k$ such that for each point $\mathbf{f}(k) = (f_1, \dots, f_k) \in \mathbf{U}(k)$ the variety defined in X by the system of equations $f_1 = \dots = f_k = 0$ is smooth and irreducible.*

A proof of the Lefschetz theorem can be found in [Hart, Theorem 8.18]

Theorem 2.23 (A version of Hodge Index Theorem)**.** *Let X be a smooth (quasi) affine irreducible surface and let $L_1, L_2 \in K(X)$ be very ample subspaces. Then we have $[L_1, L_2]^2 \geq [L_1, L_1][L_2, L_2]$.*

In the section 17 we give a proof of Theorem 2.23 using only the isoperimetric inequality for planar convex bodies and Hilbert theory for degree of subvarieties in a projective space.

Theorem 2.24 (Algebraic analogue of Alexandrov–Fenchel inequality)**.** *Let X be an irreducible smooth n-dimensional (quasi) affine variety and let $L_1, \dots, L_n \in K(X)$ be very ample subspaces. Then the following inequality holds*

$$[L_1, L_2, L_3 \dots, L_n]^2 \geq [L_1, L_1, L_3 \dots, L_n][L_2, L_2, L_3 \dots, L_n].$$

Proof. Consider n-tuples $(L_1, L_2, L_3, \dots, L_n)$, $(L_1, L_1, L_3, \dots, L_n)$ and $(L_2, L_2, L_3, \dots, L_n)$ of elements of the semigroup $K(X)$. According to Lefschetz theorem and Theorem 2.12 there is an $(n-2)$-tuple of functions $f_3 \in L_3, \dots, f_n \in L_n$ such that the system $f_3 = \dots + f_n = 0$ is non-degenerate, and defines an irreducible surface $Y \subset X$, for which the following equalities hold

$$[L_1, L_2, L_3, \dots, L_n] = [L_1, L_2]_Y,$$
$$[L_1, L_1, L_3, \dots, L_n] = [L_1, L_1]_Y,$$
$$[L_2, L_2, L_3, \dots, L_n] = [L_2, L_2]_Y.$$

By Theorem 2.23,

$$[L_1, L_2]_Y^2 \geq [L_1, L_1]_Y [L_1, L_2]_Y,$$

which proves the theorem. □

3. Semi-groups of integral points and convex bodies

3.1. **Convex bodies and their stretch ratio.** One may expect that the number of integral points in a convex body $\Delta \subset \mathbb{R}^n$ with large enough volume has the same order of magnitude as its volume. The following example show that it is not always true.

Example 3.1. Define a convex body $\Delta \subset \mathbb{R}^n$ by the following inequalities: $1/2 \leq x_1 \leq 3/4$, $0 \leq x_2 \leq a, \dots, 0 \leq x_n \leq a$. There is no integral point in Δ. But the volume of Δ equals to $(1/4)a^{n-1}$ and can be as big as one wishes.

In this section we will define the stretch ratio of a convex body and discuss its properties. In the next section we will show that if the stretch ratio of a sequence of convex bodies is bounded from above and if their volumes tend to infinity then the number of integral points in a convex body in this sequence is asymptotically equal to the its volume.

We will measure volume in $\mathbb{R}^n$ with respect to the standard Euclidian metric. Let $\Delta \subset \mathbb{R}^n$ be a bounded n-dimensional convex body. Let D be its diameter and R the radius of a largest ball which can be inscribed in Δ. In this section $B \subset \mathbb{R}^n$ will denote the unite ball centered at the origin.

Definition 3.2. The *stretch ratio* of a convex body $\Delta \subset \mathbb{R}^n$ is D/R and will be denoted by $\mu(\Delta)$.

For any $r \geq 0$ let Δ_r be the set of points a, such that a ball of the radius r centered in a is contained in Δ (in other words Δ_r consists of points inside Δ, for which the distant to the boundary of Δ is bigger than or equal to r).

Proposition 3.3. *1) For $0 \leq r \leq R$ the set Δ_r is non-empty and convex. 2) For every $b \in \Delta$, there is a point $a \in \Delta_r$, such that the distance from a to b is not bigger than $r \cdot \mu(\Delta)$.*

Proof. 1) For $0 \leq r \leq R$ the set Δ_r contains the center O of the largest ball inscribed in Δ and so is non-empty. Let $a_1, a_2 \in \Delta_r$. The set Δ contains balls $a_1 + rB$ and $a_2 + rB$. Because the set Δ is convex, it has to contain the ball $ta_1 + (1-t)a_2 + rB$ for $0 \leq t \leq 1$. So the body Δ_r contains the segment $ta_1 + (1-t)a_2$ which proves that Δ_r is convex. 2) Take a point $b \in \Delta$ and a ball of radius R centered at O which lies in Δ. One easily sees that for each $0 \leq \lambda \leq 1$ the ball of radius λR centered at the point $b - \lambda(O - b)$ also belongs to the convex body Δ. Plugging $\lambda = r/R$, and noting that the length of the vector $(O - b)$ is smaller than the diameter D of the body Δ, we get the required result. $\square$

The body Δ_r (constructed out of Δ) behaves well with respect to the Minkowski sum of convex bodies in the following sense. Let Δ_1, Δ_2 be convex bodies, let R_1, R_2 be the biggest radii of balls which could be inscribed in those bodies respectively and let $\Delta = \Delta_1 + \Delta_2$.

Corollary 3.4. *For $r_1 \leq R_1$, $r_2 \leq R_2$ we have the following inclusions:*

$$(\Delta_{1,r_1} + \Delta_{2,r_2}) + (r_1 + r_2)B \subseteq \Delta \subseteq (\Delta_{1,r_1} + \Delta_{2,r_2}) + (r_1\mu(\Delta_1) + r_2\mu(\Delta_2))B.$$

Proof. We know that

$$\Delta_{1,r_1} + r_1 B \subseteq \Delta_1 \subseteq \Delta_{1,r_1} + r_1\mu(\Delta_1)B,$$
$$\Delta_{2,r_2} + r_2 B \subseteq \Delta_2 \subseteq \Delta_{2,r_2} + r_2\mu(\Delta_2)B.$$

To get the claim it is enough to sum up the above inclusions. $\square$

Corollary 3.5. *With notations as in Corollary 3.4, the set $(\Delta_{1,r_1} + \Delta_{2,r_2})$ contains the set Δ_ρ, where $\rho = (r_1\mu(\Delta_1) + r_2\mu(\Delta_2))$.*

Proof. Follows from the inclusion

$$\Delta \subseteq (\Delta_{1,r_1} + \Delta_{2,r_2}) + (r_1\mu(\Delta_1) + r_2\mu(\Delta_2))B.$$

$\square$

We will need an estimate of the volume of the set $\Delta \setminus \Delta_r$. (We do not assume that the set Δ_r is not empty).

Theorem 3.6. *Given $r \geq 0$, for every bounded n-dimensional convex body $\Delta \subset \mathbb{R}^n$ the volume of the set $\Delta \setminus \Delta_r$, is not bigger than $(n-1)$-dimensional volume $V_{n-1}(\partial\Delta)$ of the boundary $\partial\Delta$, multiplied by r.*

Proof. We will assume that the body Δ has the smooth boundary $\partial\Delta$. This assumption is not restricted because each bounded convex body could be approximated by convex bodies with smooth boundaries. At each point $x \in \partial\Delta$ we fix a unite normal vector $\mathbf{n}_x$ looking out of the domain Δ. Consider the Riemannian manifold $\partial\Delta \times R$ — the product of the manifold $\partial\Delta$ equipped with the metric induced from $\mathbb{R}^n$ and the line $\mathbb{R}$. Consider the map $F : \partial\Delta \times R \to \mathbb{R}^n$ defined by $(x, t) \mapsto x + t\mathbf{n}_x$. Let $R_1(x) \leq \cdots \leq R_{n-1}(x)$ be the radii of the curvature of the hyper surface $\partial\Delta$ at the point x. It is easy to compute that the Jacobian $J(x, t)$ of the map F at the point (x, t) is equal to $(R_1(x) - t) \dots (R_{n-1}(x) - t)/R_1(x) \dots R_{n-1}(x)$. We call the domain $U = \{(x, t) \mid 0 \leq t < R_i(x),\ i = 1, \dots, n-1\} \subset \partial\Delta \times R$, the *regular strip*. At the points of the regular strip the Jacobian J ia positive and does not exceeded 1. Let $\Sigma \subset \Delta$ be the set of critical values of F. Let us show that each point in the set $\Delta \setminus \Sigma$ is an image, under the map F, of a point from the regular strip U. For each point $a \in \Delta$, the minimum $t(a)$ of the distance of a to the boundary $\partial\Delta$ is attained at some point $x(a) \in \partial\Delta$. The point a could be represented in the form $a = x(a) + t(a)\mathbf{n}_{x(a)}$ where $0 < t(a) < R_1(x(a))$, otherwise the point $x(a)$ is not a local minimum for the distance of a the boundary. Thus a is the image of $(x(a), t(a)) \in U$ under the map F. Denote by U_r the subset in the regular strip U, defined by the inequalities $0 \leq t < \min(r, R_i(x))$, $i = 1, \dots, n-1$. The above arguments show that each point in $\Delta \setminus \Delta_r$ which is not a critical value of the map F, belongs to the image under the map F of the set U_r. The theorem now follows by observing 1) the volume of the domain U_r is not bigger than the number $rV_{n-1}(\partial\Delta)$, 2) The Jacobian of the map F in the domain U_r is positive and does not exceed 1 and 3) the set Σ of critical values of the map F has zero measure by Sard's theorem. □

3.2. **Integral points in a convex body and its stretch ratio.** Consider a convex body having a large enough volume and assume that its stretch ratio is less than some given constant. In this section we will show that the number of integral points in such a body is, approximately, equal to the volume of the body, and an integral of a polynomial f over such a body is, approximately, equal to the sum of values of the polynomial over all integral points which belong to the convex body.

For each $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, consider the unit cube $K_a = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i < a_i + 1,\ i = 1, \dots, n\}$. These unit cubes partition $\mathbb{R}^n$.

Proposition 3.7. *Let $\Delta \subset \mathbb{R}^n$ be a bounded measurable set. Let N_1 (respectively N_2) be the number of the sets $K_{\mathbf{a}}$ which lie in Δ (respectively intersect Δ but do not lie in Δ). Then the volume $V(\Delta)$ and the number $\#(\Delta \bigcap \mathbb{Z}^n)$ of integral points belonging to Δ satisfy the following inequalities:*

1. $N_1 \leq V(\Delta) \leq N_1 + N_2$,
2. $N_1 \leq \#(\Delta \bigcap \mathbb{Z}^n) \leq N_1 + N_2$.

Proof. For a finite subset $A \subset \mathbb{Z}^n$ put $K_A = \bigcup_{a \in A} K_a$. The number of points in A is equal to the volume of the set K_A as well as the number of integral points in it. Given a bounded measurable set Δ let $A_1 = \{a \in \mathbb{Z}^n \mid K_a \subseteq \Delta\}$ and $A_2 = \{a \in \mathbb{Z}^n \mid K_a \bigcap \Delta \neq \emptyset$ but $K_a \not\subseteq \Delta\}$. Let $A = A_1 \bigcup A_2$. The number of integral points in the sets K_{A_1}, K_{A_2} and K_A are equal to N_1, N_2 and $N_1 + N_2$ respectively. By definition we have $K_{A_1} \subseteq \Delta \subseteq K_A$. The proposition now follows because the volume and the number of integral points are monotone with respect to inclusion. □

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. For a measurable set Δ we will denote the integral $\int_\Delta f(x)dx$ by $\int_\Delta f$ and the sum $\sum_{x \in \Delta \bigcap \mathbb{Z}^n} f(x)$ by $\sum_\Delta f$.

Proposition 3.8. *Let $M(f, \Delta)$ (respectively $M(df, \Delta)$) be the maximum of $|\nabla f|$ (respectively $|df|$) on Δ. The following inequalities hold:*

$$|\int_{K_{A_1}} f - \sum_{K_{A_1}} f| \leq n^{1/2} M(df, \Delta) N_1,$$

$$|\int_{\Delta} f - \sum_{\Delta} f| \leq 2M(f, \Delta) N_2.$$

Proof. The first inequality follows from Mean Value Theorem, that is, for $x, y \in K_a$, $|f(x) - f(y)|$ does not exceed the diameter of K_a ($=n^{1/2}$) multiplied by the maximum of $|\nabla f|$. Second inequality follows from the inequalities $|\int_\Delta f| \leq M(f, \Delta)N_2$, and $|\sum_\Delta f| \leq M(f, \Delta)N_2$. □

Corollary 3.9. $|\int_\Delta f - \sum_\Delta f| \leq n^{1/2} M(df, \Delta) N_1 + 2M(f, \Delta) N_2$.

Proposition 3.10. *Let $\Delta \in \mathbb{R}^n$ be a convex body contained in a ball of the radius D. Then the number N_2 of the sets $K_{\mathbf{a}}$ which intersect Δ but do not belong to Δ satisfy the inequality*

$$N_2 \leq N_2(D, n) = 2n^{1/2} \omega(n-1)(D + n^{1/2})^{n-1},$$

where $\omega(n-1)$ is the $(n-1)$-dimensional volume of the unite $(n-1)$-dimensional sphere.

Proof. As above let K_{A_2} be the union of the sets K_a, which intersect Δ but not lie in it. Because the diameter of the unite cube is $n^{1/2}$ we have $\Delta_r \subseteq K_{A_2} \subseteq \Delta + rB$ where $r = n^{1/2}$. According Theorem 3.6 the volume of the set $(\Delta + rB) \setminus \Delta_r$ does not exceed the number $V_{n-1}(\Delta + rB)2r$. The convex body $\Delta + rB$ is contained in a ball of the radius $D + r$ and hence $V_{n-1}(\Delta + rB) \leq \omega_{(n-1)}(D + r)^{n-1}$ (note that if a convex body Δ_1 is contained in another convex body Δ_2 then $V_{n-1}(\partial \Delta_1) < V_{n-1}(\partial \Delta_2)$). Thus $N_2 = V_n(K_{A_2}) \leq N_2(D, n)$. □

Proposition 3.11. *Let Δ be a convex body with diameter is D which contains a ball of radius R. Then the volume of Δ is bigger than or equal to*

$$V(D, R, n) = DR^{n-1}\Omega_{n-1}/2(n-1)!$$

, *where* Ω_{n-1} *is the volume of the unite* $(n-1)$*-dimensional ball.*

Proof. Let O be the center of a ball of radius R contained in Δ. Since the diameter of Δ is D, there is a point $b \in \Delta$ whose distance from O is bigger than or equal to $D/2$. Then Δ contains the cone of revolution whose apex is b, its base is an $(n-1)$-dimensional ball of radius R centered at O and its height equal to $D/2$. The volume of this cone is $V(D, R, n)$. □

Corollary 3.12. *Let* $\Delta \subset \mathbb{R}^n$ *be a convex body with volume* $V(\Delta)$ *and the stretch ratio* $\mu(\Delta)$*. If* Δ *contains a unite ball then*

$$N_2/V(\Delta) \leq F(\mu(\Delta), n)V(\Delta)^{-1/n},$$

for an explicitly defined function F*.*

Proof. Using the estimates in Propositions 9.4 and 9.5 one see that, up to explicitly computable constants, the quantity $N_2/V(\Delta)$ can be estimated from above by the expression

$$(D + n^{1/2})^{n-1}/R^{n-1}D.$$

Using the relations $R > 1$, $V^{1/n}(\Delta) < D$ and $\mu(\Delta) = D/R$ one obtains

$$(N_2/V(\Delta) \leq \mu(\Delta) + n^{1/2})^{n-1}V(\Delta)^{-1/n}.$$

□

Remark 3.13. Since $D > V^{1/n}(\Delta)$ we have $R > V^{1/n}(\Delta)/\mu(\Delta)$. So if the volume of Δ is bigger than $\mu(\Delta)^n$, then Δ automatically contains a unite ball and we can drop the condition of containing a unit ball in Corollary 3.12 for such convex bodies.

Let Δ be a bounded convex n-dimensional body. Denote the multiplication of Δ by a scalar $\lambda > 0$ with $\lambda\Delta$. The following relations hold

$$V(\lambda\Delta) = \lambda^n V(\Delta),$$
$$V_{n-1}(\partial(\lambda\Delta)) = \lambda^{n-1}V_{n-1}(\partial(\Delta)).$$

Let $f : \mathbb{R}^n \to R$ be a homogeneous C^1 function of degree $\alpha \geq 0$, i.e. $f(\lambda x) = \lambda^\alpha f(x)$. From homogeneity of f we have:

$$M(f, \lambda\Delta) = \lambda^\alpha M(f, \Delta),$$
$$M(df, \lambda\Delta) = \lambda^{\alpha-1}M(f, \Delta),$$
$$\int_{\lambda\Delta} f(x)dx = \lambda^{\alpha+n}\int_\Delta f(x)dx.$$

Theorem 3.14. *Let* $\Delta \subset \mathbb{R}^n$ *be a bounded* n*-dimensional convex body and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a homogeneous* C^1 *function of the degree* $\alpha \geq 0$*. Then*

$$\lim_{\lambda\to\infty} \frac{\sum_{x\in\lambda\Delta\cap\mathbb{Z}^n} f(x)}{\lambda^{\alpha+n}} = \int_\Delta f(x)dx.$$

Proof. From Corollary 3.9 we have

$$\frac{|\int_{\lambda\Delta} f(x)dx - \sum_{x\in\lambda\Delta\cap\mathbb{Z}^n} f(x)|}{\lambda^{\alpha+n}} \leq \frac{n^{1/2}M(df,\lambda\Delta)}{\lambda^\alpha}\cdot\frac{N_1(\lambda\Delta)}{\lambda^n}+\frac{2M(f,\lambda\Delta)}{\lambda^\alpha}\cdot\frac{N_2(\lambda\Delta)}{\lambda^n}.$$

As $\lambda \to \infty$, the expressions $\frac{N_1(\lambda\Delta)}{\lambda^n} \leq V(\Delta)$ and $\frac{2M(f,\lambda\Delta)}{\lambda^\alpha} = 2M(f,\Delta)$ remain bounded but $\frac{n^{1/2}M(df,\lambda\Delta)}{\lambda^\alpha}$ tends to 0 (if $\alpha = 0$ the function f is constant and the last term vanishes) and $\frac{N_2(\lambda\Delta)}{\lambda^n} \to 0$ (see Corollary 3.12). This proves the theorem. □

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree k and let $f = f_0 + f_1 + \cdots + f_k$ be its decomposition into homogeneous components.

Corollary 3.15.

$$\lim_{\lambda\to\infty} \frac{\sum_{x\in\lambda\Delta\cap\mathbb{Z}^n} f(x)}{\lambda^{n+k}} = \int_\Delta f_k(x)dx.$$

Proof. According to Theorem 3.14, for any $0 \leq i \leq k$ we have

$$\lim_{\lambda\to\infty} \frac{\sum_{x\in\lambda\Delta\cap\mathbb{Z}^n} f_i(x)}{\lambda^{n+i}} = \int_\Delta f_i(x)dx.$$

The corollary easily follows from this. □

Corollary 3.16. *Let Δ be a bounded convex body. Then*

$$\lim_{\lambda\to\infty} \frac{\#(\lambda\Delta \bigcap \mathbb{Z}^n)}{\lambda^n} = V(\Delta).$$

Proof. Apply Theorem 3.14 to $f \equiv 1$. □

3.3. **Semigroups of integral points.** The set of points $(h, \mathbf{x})$ in $\mathbb{R} \times \mathbb{R}^n$ with $h \geq 0$ is called the *positive half-space*. We call a semigroup $G \subset \mathbb{Z}\times\mathbb{Z}^n$ a *graded semigroup*, if the following conditions are satisfied: 1) G is contained in the positive half-space. 2) For each $d \in \mathbb{N}$ the set of elements of G of the form $(d, \mathbf{m})$ is non-empty. We will say that an element $(d, \mathbf{m}) \in G$ has *degree d*.

Now we define the class of semigroups G which will play a key role in for us. We need the following two definitions:

1) A closed convex $(n + 1)$-dimensional cone C in the positive half-space is called a *positive cone* if its intersection with the horizontal hyperplane $h = 0$ contains only the origin.

2) To an integral point $A = (1, \mathbf{x})$, with the first coordinate equal to 1, and a subgroup $T \subset \mathbb{Z}^n$, we associate the subgroup $lA + T \subset \mathbb{Z} \times \mathbb{Z}^n$ of vectors $lA+\mathbf{x}$, where $l \in \mathbb{Z}$, $\mathbf{x} \in T$. Obviously if $A_1 - A_2 \in T$, the subgroups $lA_1 + T$ and $lA_2 + T$ coincide.

Definition 3.17. Let C be a positive cone, T a subgroup of a finite index in $\mathbb{Z}^n$, and A an integral point with the first coordinate equal to 1 (the point A is defined up to addition of an element from the group T). We say that a semigroup $G \subset \mathbb{Z} \times \mathbb{Z}^n$ has type (C, T, A) if $G = C \cap lA \times T$.

The following statement is clear.

Proposition 3.18. *Two semigroups of types* (C_1, T_1, A_1) *and* (C_2, T_2, A_2) *coincide if and only if* $C_1 = C_2$, $T_1 = T_2$ *and the difference* $A_1 - A_2$ *belongs to the group* $T_1 = T_2$.

And a few extra definitions. Let G be a graded semi-group.

(1) G has *finite sections*, if for every $d > 0$ the set of elements of degree d in G is finite. We will denote the number of elements of degree d by $H_G(d)$. We call H_G the *Hilbert function* of a graded semigroup with the finite sections.
(2) G has *conic type*, if it is contained in a positive cone.
(3) G has *limited growth*, if it has finite sections and $H_G(d) < qd^n$, for a constant q.
(4) G has *complete rank*, if the subgroup $\mathbb{Z} \times \mathbb{Z}^n$, generated by the semi-group G, has finite index in $\mathbb{Z} \times \mathbb{Z}^n$.
(5) G is *saturated*, if the subgroup generated by the semigroup G is the whole $\mathbb{Z} \times \mathbb{Z}^n$.

It is clear that if a semi-group has conic type then it has a limited growth. Also if it is saturated, then it has complete rank.

We are interested in semigroups with finite sections, complete rank and limited growth. We will see that one can obtain a more or less complete description asymptotic behavior of such semigroups.

Suppose a semigroup G is contained in a semigroup M of the type (C, T, A). Denote the sections of G, M and the cone C by the hyperplane $h = d$ respectively by $G(d)$, $M(d)$ and $C(d)$. Consider the function $r_{G,M}(d)$ defined as the minimum distance from a point in the set $M(d) \setminus G(d)$ to the boundary of the section $C(d)$.

Definition 3.19. A semigroup $M \subset \mathbb{Z} \times \mathbb{Z}^n$ of a type (C, T, A) *approximates the semigroup* G if:

(1) The semigroup M contains the semigroup G.
(2) We have
$$\lim_{d \to \infty} \frac{r_{G,M}(d)}{d} = 0$$

The condition 2) is equivalent to the following condition: 2'): there exists a function $P(\rho)$ such that for all $d\mathbb{N}$ and $\rho > 0$ we have $r_{G,M}(d) < \rho d + P(\rho)$.

The following statement is clear.

Proposition 3.20. *Let* $G_1 \subseteq G_2 \subset \mathbb{Z} \times \mathbb{Z}^n$ *be semigroups which are approximated by the semigroups* M_1, M_2 *of the types* (C_1, T_1, A_1) *and* (C_2, T_2, A_2) *respectively. Then* $C_1 \subseteq C_2$, $\Delta(G_1) \subseteq \Delta(G_2)$ *and* $T_1 \subseteq T_2$. *The point* A_2 *can be chosen equal to* A_1.

The following Theorem 3.21 will be important for us. Let A be a finite subset in $\mathbb{Z}^n \subset \mathbb{R}^n$, and let T be the subgroup of $\mathbb{Z}^n$ generated by A. Denote

the convex hull of A by $\Delta \subset \mathbb{R}^n$. Let $k * A$ be the set $\underbrace{A + \cdots + A}_{k \text{ times}}$, which consists of all sums of k-tuples $a_1, \ldots, a_k$ of elements of the set A.

Theorem 3.21 ([Khov1]). *Let $T \subset \mathbb{Z}^n$ be a subgroup of finite index. Then there is a constant P (independent on k) such that every point in $k\Delta \cap T$ whose distance to the boundary $\partial(k\Delta)$ of the polyhedron is not smaller than P belongs to $k * A$.*

Theorem 3.22. *Let $G \subset \mathbb{Z} \times \mathbb{Z}^n$ be a semigroup of complete rank and with finite sections. If G has limited growth then it is of conic type.*

Proof. The idea of the proof is as follows. If G does not belong to any positive cone, then given any constant L one can find a certain sequence d_i of natural numbers such that the Hilbert function $H_G(d_i)$ is bigger than Ld_i^n. Proof is based on Theorem 3.21. To start we need some auxiliary constructions.

Let A be any point in $G(1)$ (recall that for any $i \in \mathbb{N}$, $G(i)$ is non-empty). Consider the sets $G(d) - dA$ considered as subsets of the coordinate hyperplane $h = 0$. They possess the following properties: 1) The origin belongs to each set $G(d) - dA$, 2) $(G(d_1) - d_1A) + (G(d_2) - d_2A) \subseteq G(d_1 + d_2) - (d_1 + d_2)A$, 3) If $d_1 \leq d_2$ then $(G(d_1) - d_1A) \subseteq (G(d_2) - d_2A)$, 4) The union of sets $G(d) - dA$ generates a subgroup T of finite index in $\mathbb{Z}^n$, 5) For $k \gg 0$ the set $G(k) - kA$ generates the subgroup T (because the subgroup T is finitely generated).

Denote by $\Delta(d)$ the convex hull of the set $G(d)$. For $k \gg 0$ the set $G(k)$ generates the group T which has complete rank and hence the polyhedron $\Delta(k)$ contains n linearly independent vectors. Fix a k_0 for which this is the case. Consider the polyhedron $\frac{1}{k_0}\Delta(k_0)$ located in the plane $h = 1$. One can find an n dimensional ball of a radius $R > 0$ in this polyhedron. Let $(1, \tilde{O})$ be the center of this ball. Let us show that for each point $(m, \mathbf{x}) \in G$ the distance l from the points $(1, \frac{\mathbf{x}}{m})$ to $(1, \tilde{O})$ can be estimated from above. In fact, the n dimensional volume V of a convex body, which contains a ball of the radius R and a point whose distance to the center of the ball is equal to l has to be bigger than or equal to $V = cR^{n-1}l/(n-1)!$, where c is the volume of the unite $(n-1)$ dimensional ball. In the next paragraph we will show that there is a sequence d_i of arguments, such that the limit of $H_G(d_i)/d_i^n$, as $i \to \infty$, is bigger than or equal to V/I where I is the index of the semigroup T in $\mathbb{Z}^n$. From the assumption $H(G, d_i) < Ld_i^n$ we see that $V/I \leq L$ and that $l \leq l_0 = IL(n-1)/c(n-1)R^{n-1}$ which give an estimate of the distance l. Now the semigroup G belongs to the positive cone C, whose section $C(1)$, by the hyperplane $h = 1$, is the ball of the radius l_0 centered at $(1, \tilde{O})$. This shows that G has conic type.

Now let us show how to construct the sequence $\{d_i\}$. Take k_0 and $(m, \mathbf{x})$ as above. Then the convex hull of the section $G(k_0)$ projected to the plane $h = 1$ has volume bigger than or equal to V. Let $d = k_0m$. The convex

hull of the section $G(d)$ contains $d\Delta$ whose volume is greater than or equal to Vd^k. The points in the section $G(d)$ generate a subgroup $T \in \mathbb{Z}^n$ of the index I. For $i \in \mathbb{N}$ put $d_i = id = ik_0m$. Now applying Theorem 3.21 to the set $G(d)$ we see that, for large enough i, $H(G, d_i)/d_i^n$ can not be smaller than V/I. This finishes the proof of the theorem. $\square$

Theorem 3.23. *Let $G \subset \mathbb{Z} \times \mathbb{Z}^n$ be a semigroup of complete rank, with finite sections. If the semigroup has a limited growth then the semigroup G then there is a semigroup M of the type (C, T, A), which approximates the semigroup G. Such semigroup M is unique.*

Proof. The proof of Theorem 3.23 resembles the proof of Theorem 3.22. Both of them are based on Theorem 3.21. Denote by $G(\leq d)$ the finite subset in the semigroup G consisting of all the elements with degrees not bigger than d. Let $\tilde{G}(\leq d)$ be the projection of the set $G(\leq d)$ from the origin to the hyperplane $h = 1$, (i.e. if $(m, \mathbf{x}) \in G(\leq d)$, then $(1, \frac{\mathbf{x}}{m}) \in \tilde{G}(\leq d)$). Let $\tilde{\Delta}(\leq d)$ be the convex hall of $\tilde{G}(\leq d)$. In Theorem 3.22 we obtained a increasing sequence of the convex bodies $\tilde{\Delta}(\leq 1) \subseteq \ldots, \subseteq \tilde{\Delta}(\leq d) \subseteq \ldots$ all contained in a bounded convex body. Let $\Delta = \bigcup_{1 \leq q} \Delta(\leq d)$ and $\bar{\Delta}$ be its closure. We will show that the semigroup M of the type (C, T, A) approximates the semigroup G, where:

$C =$ the positive cone, whose section by the hyperplane $h = 1$ coincides with $\bar{\Delta}$.

$T =$ the intersection of the subgroup in $\mathbb{Z} \times \mathbb{Z}^n$, generated by the semigroup G with the group $\mathbb{Z}^n = \{0\} \times \mathbb{Z}^n$,

$A =$ any element of degree one in the semigroup G, $A \in G(1)$.

Fix a $\rho > 0$ and a positive cone $C_\rho \subset C$ such that its section $C_\rho(1)$ by the hyperplane $h = 1$ lies strictly inside the section $C(1)$ of C, and such that the distance from $C_\rho(1)$ to the boundary of $C(1)$ is greater than ρ. To proof the theorem it is enough to show that given the cone C_ρ there is a constant $P(\rho)$ (independent on k) such that any point inside the section $C_\rho(k)$ whose distance to boundary $\partial(C_\rho(k))$ is bigger than or equal to P_ρ and which is representable in the form $kA + T$, belongs to the semigroup G.

For $K > k$ regard $G(k)$ as a subset of $G(K)$ by adding the vector $(K - k)A \in G$ to all the points in $G(k)$. Fix any (small) positive number ρ. The increasing sequence of the convex bodies $\tilde{\Delta}(\leq d)$ converges to the body $\bar{\Delta}$. So starting from some number d_1 the Hausdorff distance between the bodies $\tilde{\Delta}(\leq d)$ and $\bar{\Delta}$ is smaller than ρ. Then starting from some number d_2 the set $G(\leq d_2)$ and the semigroup G generate the same subgroup. Let $m \in \mathbb{N}$ be bigger than d_1 and d_2. Consider the section $G(m!)$ of the semigroup G. It has the following properties:

1) The projection of the section $G(m!)$ from the origin to the hyperplane $h = 1$ contains the set $\tilde{G}(\leq q_0)$. In fact if $(p, \mathbf{x}) \in G(\leq q_0)$ then $\frac{m!}{p}(p, \mathbf{x}) \in G(m!)$ because G is a semigroup and the number $m!$ is divisible by the number $p \leq m$.

2) The differences of the points in the section $G(m!)$ generate the group T. Because the points of the section $G(\leq d_2)$ could be shifted to the section $G(m!)$ by adding the vector kA for an appropriate k. By the assumption the intersection of the group generated by the set $G(\leq d_2)$ with the horizontal hyperplane is equal to T. So the differences of the points on in $G(m!)$ generates the group T.

Now let us apply Theorem 3.21 to the section $G(m!)$ and the sums

$$\underbrace{G(m!) + \cdots + G(m!)}_{k \text{ times}},$$

which belong to $G(km!)$. Let $T(A, P, km!)$ be the subset of the group $lA+T$ consisting of the points in the set $km!\tilde{\Delta}(\leq m!)$ such that their distance to boundary of this polyhedra is bigger than P. According to Theorem 3.21 there is a constant P, such that for each k any point in the set $T(A, P, km!)$ belongs to the semigroup G. Thus we may find many points from the group $lA + T$ in the sections $G(d)$ of the semigroup G where d is divisible by $m!$. Now let d be equal to $km! + q$ with $0 \leq q < m!$. The section $G(d)$ contains points of the set $T(A, P, km!)+qA$ which belong to the group $lA+T$. Denote by D the diameter of the polyhedron $\tilde{\Delta}(\leq m!)$. We show that all the points of $lA+T$ in the polyhedron $(km!+q)\tilde{\Delta}(\leq m!)$ such that their distance to the boundary of this polyhedron is bigger than $m!D + P$ are in the semigroup G. Indeed, such points are inside the polyhedron $km!\tilde{\Delta}(\leq m!) + qA$ and their distance to the boundary is bigger than or equal to P. So they are in the semigroup G. We proved that each point in $lA+T$ which belongs to the section $C(h)$ and whose distance to the boundary is bigger than $\rho h+m!D+P$ belongs to G. The theorem is now proved. $\square$

Definition 3.24. Consider a semigroup G which can be approximated by a semigroup M of the type (C, T, A). The *Newton convex body* $\Delta(G)$ of the semigroup G is the convex body obtained by the intersecting the cone C with the hyperplane $h = 1$ in the space $\mathbb{R} \times \mathbb{R}^n$. We regard $\Delta(G)$ as a subset of $\mathbb{R}^n$.

Theorem 3.25. *Assume that a semigroup G can be approximate by a semigroup of the type (C, T, A). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 homogenous function of degree $\alpha \geq 0$. Then*

$$\lim_{d\to\infty} \frac{\sum_{x\in G(d)} f(x)}{d^{\alpha+n}} = \frac{1}{ind(T)} \int_{\Delta(G)} f(x)dx,$$

where $ind(T)$ is the index of subgroup $T \subset \mathbb{Z}^n$, $G(d)$ is the set of elements of degree d in G and $\Delta(G)$ is the Newton convex body of the semigroup G.

Proof. If the index of the subgroup $T \in \mathbb{Z}^n$ is equal to 1, the theorem follows from Theorem 3.14. If the index is bigger than 1, one can make a linear change of variables and transform the subgroup T into the whole lattice $\mathbb{Z}^n$. Such change of variable changes the volume by the factor $1/ind(T)$. Also

$1/ind(T)$ is responsible for the asymptotical behavior of the sum of values of f on the degree d elements of the semigroup as $d \to \infty$. $\square$

Corollary 3.26. *Assume that a semigroup G can be approximate by a semigroup of the type (C, T, A). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let $f = f_0 + f_1 + \cdots + f_k$ be its decomposition into homogenous components. Then*

$$\lim_{d\to\infty} \frac{\sum_{x\in G(d)} f(x)}{d^{n+k}} = \frac{1}{ind(T)} \int_{\Delta(G)} f_k(x)dx.$$

Corollary 3.27. *Assume that a semigroup G can be approximated by a semigroup of the type (C, T, A). Then the Hilbert function H_G has the following asymptotical behavior:*

$$\lim_{d\to\infty} \frac{H_G(d)}{d^n} = \frac{1}{ind(T)} V(\Delta(G)),$$

where $V(\Delta(G))$ is the n-dimensional volume of the Newton convex body $\Delta(G)$ of the semigroup G.

Assume that a graded semigroup G is contained in another graded semigroup $G_1 \subset \mathbb{Z} \times \mathbb{Z}^n$ of complete rank and with limited growth. For such semigroups all properties we are interested in are corollaries of the results proved above. Let us discuss this in more details: let $A \in G(1)$ be a degree 1 element in G. Denote by T the intersection of the subgroup generated by G and $\mathbb{Z}^n = \{0\} \times \mathbb{Z}^n$. Assume that the group T has rank k. Consider the subgroup $M \subset \mathbb{Z}^n = \{0\} \times \mathbb{Z}^n$ consisting of all the elements m which after multiplication by a natural number $l(m)$ lie in T i.e. $l(m)m \in T$. The group M is isomorphic to $\mathbb{Z}^k$ and after a choice of a basis can be identified with this group. The group T is a subgroup of a finite index in M. The group generated by the semigroup G is contained in the group $\langle A \rangle \times M \simeq \mathbb{Z} \times \mathbb{Z}^k$ generated by A and $\{0\} \times M$. By the assumption the semigroup G is contained in a graded semigroup of complete rank and with limited growth. So the semigroup G is contained in a closed positive cone C (Theorem 3.22). Let us call the vector space generated by the group $\langle A \rangle \times M$ the *space of the semigroup* G and denote it by $V(G)$. It is isomorphic to $\mathbb{R} \times \mathbb{R}^k$. The group $\langle A \rangle \times M$ is a lattice in this space and defines a Euclidean metric in the space of the semi-group G in which the volume of the parallelepiped given by the generators of $\langle A \rangle \times M$ is equal to 1. Semigroup G belongs to the positive closed cone $C_1 = C \cap \mathbb{R} \times \mathbb{R}^k$ in this space. Now one can apply Theorem 3.25 to the semigroup G.

Definition 3.28. Let $C(G)$ denote the closure of the convex hull of $G \cup \{0\}$. The intersection of $C(G)$ with the horizontal hyperplane $h = 1$, namely $\Delta(G) = C(G) \cap \{h = 1\}$, will be called The *Newton convex body of the semigroup* G.

Let $G \in \mathbb{Z} \times \mathbb{Z}^n$ be a graded semigroup which is contained in a graded semigroup $G_1 \subset \mathbb{Z} \times \mathbb{Z}^n$ of complete rank and with limited growth. Assume that the rank of the group $T(G)$ is equal to k. We have the following:

Theorem 3.29. *The Newton convex body $\Delta(G)$ of the semigroup G is a bounded k-dimensional convex body. In the space $V(G)$ of the semigroup G, $C(G)$ is a convex cone of maximum dimension and G can be approximated by a semigroup of type $(C(G), T, A)$ in $V(G)$. Finally the Hilbert function H of the semigroup G has the following asymptotical behavior:*

$$\lim \frac{H(d)}{d^k} = \frac{k! V_k(\Delta(G))}{ind T(G)},$$

where V_k is the k-dimensional volume and $ind(T)$ is the index of the subgroup T in M.

3.4. **Addition of graded semigroups.** Let us start with a lemma about sum of subgroups in the lattice $\mathbb{Z}^n$.

Lemma 3.30. *Let T_1, T_2 be subgroups of finite index in $\mathbb{Z}^n \subset \mathbb{R}^n$ and let T be the sum of these subgroups. Then there exists a number N with the following property: for any representation of an element $a \in T$ in the form $a = \mathbf{x} + \mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, one can find elements $b \in T_1$ i $c \in T_2$ such that $a = b + c$ and $||a - \mathbf{x}|| < N$, $||b - \mathbf{y}|| < N$.*

Proof. Fix generators $p_1, \dots, p_n$ in the group T_1. The vectors $p_1, \dots, p_n$ form a basis in $\mathbb{R}^n$. Each vector $\mathbf{u} \in \mathbb{R}^n$ can be represented in the form $\lambda_1 p_1 + \dots + \lambda_n p_n = Q_1 + M_1$, with $Q_1 = [\lambda_1]p_1 + \dots + [\lambda_n]p_n$ belongs to T_1, and $M_1 = (\lambda_1 - [\lambda_1])p_1 + \dots + (\lambda_n - [\lambda_n])p_n$ has a length not bigger than D_1 where $D_1 = \sum ||p_i||$. 2) In a similar way each vector $\mathbf{v} \in \mathbb{R}^n$ can be represented in the form $Q_2 + M_2$, where the vector Q_2 belongs to the group T_2, and the vector M_2 has a uniformly bounded length $||M_2|| < D_2$. For each vector r_i from a finite subset in the group $T_1 + T_2$ of vectors whose length is not bigger than $D_1 + D_2$. Fix a representation in the form $r_i = b_i + c_i$ where $b_i \in T_1$ and $c_i \in T_2$. Denote by D_3 the number $D_3 = \sum ||b_i|| + \sum ||c_i||$.

We proceed with the proof as follows. Assume that an element $a \in T$ is represented in the form $a = \mathbf{x} + \mathbf{y}$. For the vector $\mathbf{x}$ (respectively $\mathbf{y}$) one can find a vector $Q_1 \in T_1$ (respectively $Q_2 \in T_2$) such that $||\mathbf{x} - Q_1|| < M_1$ (respectively $||\mathbf{y} - Q_2|| < M_2$). For a vector $r = (a - Q_1 - Q_2) \in T$ whose length is not bigger than $D_1 + D_2$ there is a representation in the form $r = b_i + c_i$ where $b_i \in T_1, c_i \in T_2$ and $||b_i||, ||c_i|| < D_3$. So we have represented the vector a in the form $a = (Q_1 + b_i) + (Q_2 + c_i)$, where the vectors $(Q_1 + b_i)$ and $(Q_2 + c_i)$ belong to the groups T_1 and T_2 respectively and the following inequalities holds

$$||\mathbf{x} - (Q_1 + b_i)|| < D_1 + D_3,$$

$$||\mathbf{y} - (Q_2 + c_i)|| < D_2 + D_3.$$

Taking $N = D_1 + D_2 + D_3$ completes the proof. □

Consider the following addition on $\mathbb{R} \times \mathbb{R}^n$ between the vectors which have the same first coordinate:

$$(h, \mathbf{x}_1) \oplus_t (h, \mathbf{x}_2) = (h, \mathbf{x}_1 + \mathbf{x}_2).$$

Equivalently $(h, \mathbf{x}_1) \oplus_t (h, \mathbf{x}_2) = (h, \mathbf{x}_1) + (h, \mathbf{x}_2) - h\mathbf{e}$ where $\mathbf{e}$ is the unit vector $(1, 0, \dots, 0) \in \mathbb{R} \times \mathbb{R}^n$. For two subsets $A, B \subset \mathbb{R} \times \mathbb{R}^n$ let $A \oplus_t B$ be the collection of all $a \oplus_t b$ where $a \in A$, $b \in B$ and a, b have the same first coordinates.

The following statement is obvious.

Proposition 3.31. *Let G_1, G_2 be graded semigroups then: 1) $G_1 \oplus_t G_2$ is a graded semigroup. 2) If G_1, G_2 have finite sections (respectively conic type) then the semigroup $G_1 \oplus_d G_2$ also has finite sections (respectively conic type).*

Let $A_1 = (1, \mathbf{x}_1)$, $A_2 = (1, \mathbf{x}_2)$ be two points in the hyperplane $\{h = 1\}$. Put $A = A_1 \oplus_t A_2 = (1, \mathbf{x}_1 + \mathbf{x}_2)$.

Theorem 3.32. *Let G_1, G_2 be graded semigroups which can be approximated by semigroups M_1,M_2 of the types (C_1, T_1, A_1) and (C_2, T_2, A_2). Then the semigroup $G_1 \oplus_t G_2$ can be approximated by a semigroup of the type (C, T, A) where $C = C_1 \oplus_t C_2$, $T = T_1 + T_2$ and $A = A_1 \oplus_t A_2$.*

Proof. By assumption the semigroups G_1 and G_2 belong to the cones C_1 and C_2 respectively and $G_1 \oplus_t G_2$ is contained in the cone $C_1 \oplus_t C_2$. Also G_1 and G_2 contain points A_1 and A_2 and their intersections with the hyperplane $h = 0$ are equal to T_1 and to T_2 respectively. So the semigroup $G_1 \oplus_t G_2$ contains the point $A = A_1 \oplus_t A_2$) and its intersection with the hyperplane $h = 0$ is $T_1 + T_2$. By the assumption the semigroups M_1 and M_2 approximate the semigroups G_1 and G_2. So: 1) Each point in the section $G_1(d) = G_1 \cap \{h = d\}$ whose distance to the boundary $C_1(d) = C_1 \cap \{h = d\}$ of the section is bigger than $r_1 = r(G_1, M_1)(d)$ has to belong to the subgroup T_1 shifted by the vector dA_1. 2) Each point in the section $G_2(d) = G_2 \cap \{h = d\}$ whose distance to the boundary $C_2(d) = C_2 \cap \{h = d\}$ of the section is bigger than $r_2 = r(G_2, M_2)(d)$ has to belong to the subgroup T_2 shifted by the vector dA_2.

Let us reformulate the statement from the above paragraph. Let us consider in the hyperplane $h = 0$: 1) the convex body $\Delta_1(d) = C_1(d) - dA_1$ and its $(\Delta_1(d))_{r_1}$ where $r_1 = r(G_1, M_1)(d)$. Each point belonging to the intersection of the set $(\Delta_1(d))_{r_1}$ and of the group T_1 belongs to the set $G_1(d) - dA_1$. 2) the convex body $\Delta_2(d) = C_2(d) - dA_2$ and its $(\Delta_2(d))_{r_2}$ where $r_2 = r(G_2, M_2)(d)$. Each point belonging to the intersection of the set $(\Delta_2(d))_{r_2}$ and of the group T_2 belongs to the set $G_2(d) - dA_2$.

Let R_1 and R_2 be the radiuses of the maximal balls in the convex bodies $\Delta_1 = C_1(1) - \mathbf{e}$ and $\Delta_2 = C_2(1) - \mathbf{e}$ in the hyperplane $h = 0$ and let D_1 and D_2 be the diameters of those bodies. The bodies Δ_1 and Δ_2 are the sections of the cones C_1 and C_2 by the hyperplane $h = 1$ shifted by adding the vector $-\mathbf{e}$ to the hyperplane $h = 0$. The similar characteristics of the

sections of the cones C_1 and C_2 by the hyperplane $h = t$ shifted by adding the vector $-t\mathbf{e}$ to the hyperplane $h = 0$ are correspondingly (tR_1, tD_1) and (tR_2, tD_2). For each cone the ratio tD_i/tR_i is independent on t.

Consider in the hyperplane $h = 0$ the convex bodies $\Delta_1(d) = C_1(d) - dA_1$ and $\Delta_2(d) = C_2(d) - dA_2$. According to the corollary 8.3 for every $r \ll d$ the Minkowski sum of the sets $(\Delta_1(d))_r$ and $(\Delta_2(d))_r$ contains the set $(\Delta(d))_{Qr}$, gde $Q = (D_1/R_1) + (D_2/R_2)$.

Take $r(d) = \max[r_1, r_2] + N$ where $r_1 = r(G_1, M_1)(d)$, $r_2 = r(G_2, M_2)(d)$. Then all points in the intersection of the body $(\Delta_1(d))_{r(d)-N}$ and of the group T_1 belong to the set $G_1(d) - dA_1$, all points in the intersection of the body $(\Delta_2(d))_{r(d)-N}$ and of the group T_2 belong to the set $G_2(d) - dA_2$. According to the corollary 8.3 each a in the intersection of the set $(\Delta(d))_{Qr(d)}$ and of the group $T_1 + T_2$ could be represented as sum of vectors $\mathbf{x}$ and $\mathbf{y}$ where $\mathbf{x} \in (\Delta_1(d))_{r(d)}$ and $\mathbf{y} \in (\Delta_2(d))_{r(d)}$. According to the lemma 11.1 instead of the vectors $\mathbf{x}$ i $\mathbf{y}$ one can use the elements b and c of the groups T_1 and T_2 which belong to the sets $(\Delta_1(d))_{r(d)-N}$ and $(\Delta_2(d))_{r(d)-N}$.

By definition $r(d)$ is big enough so the points b and c belong to the sets $G_1(d) - dA_1$ and $G_1(d) - dA_1$. So each point of the intersection of the body $(\Delta(d))_{Qr(d)}$ and of the group $T_1 + T_2$ belongs to the set $G_1(d) + G_2(d) - d(A_1 + A_2)$.

Basically the theorem is proved: to complete the proof we need some arithmetic calculations. Using the bodies $C_1(1)$, $C_2(1)$ we found the constants D_1/R_1 and D_2/R_2. Using the groups T_1 and T_2 we found the constant N. Assume now that the semigroup M_1 and M_2 approximate the semigroups G_1 and G_2 and $r_1 = r(G_1, M_1)$, $r_2 = r(G_2, M_2)$ are the functions which appeared in the definition of approximation. The function Qr, where $r = \max[r_1, r_2] + N$ is bigger than $\tilde{r} = r(G_1 \oplus_t G_2, T_1 + T_2, A_1 \oplus_1 A_2)$. Because of the relations $\lim_{d\to\infty} r_1(d)/d = \lim_{d\to\infty} r_2(d)/d = 0$ one can claim that $\lim_{d\to\infty} Qr(d)/d = 0$. So $\lim_{d\to\infty} \tilde{r}(d)/d = 0$. □

4. Convex body associated to a subspace of regular functions and main theorem

4.1. Pre-valuations, valuations and Gröbner maps. The original example of a valuation on the space of meromorphic functions in one variable is the degree of zero or pole of a function f at a given point say the origin. If f is a Laurent polynomial this is equal to the degree of the smallest non-zero term of f. In higher dimensions one can take the exponent of the smallest non-zero term of f (with respect to an ordering of terms) as a valuation. In this section we will discuss valuations on the field of rational functions on an irreducible n-dimensional (quasi) affine variety X. We will use a valuation to associate a semi-group of integral points to a subspace $L \in K(X)$ of regular functions. We will then use results of the previous sections on semi-groups to get our main result on relation between number of solutions of systems of algebraic equations on X and volume of convex

bodies. We will be interested in valuations which have values in $\mathbb{Z}^n$ and in particular faithful valuations, i.e. valuations for which all the integral points in $\mathbb{Z}^n$ appear as values of the valuation for some rational function. Classically valuations are used to prove the existence of a unique smooth model for a birational class of algebraic curves. These ideas has been generalized by great classical algebraic geometers e.g. Zariski, to attack the problem of resolution of singularities in higher dimensions (see [Hodge-Pedoe] for a classical treatment of valuation theory in algebraic geometry). We should mention that will deal with the so-called *non-Archimedean* valuations only.

Let $(I, >)$ be an ordered set. *A representation* of I in the category of vector subspaces of a vector space V is a map, which associates to each $\alpha \in I$ a non-zero subspace $V_\alpha \subseteq V$ such that if $\alpha < \beta$ then $V_\alpha \subseteq V_\beta$ and $\cup_{i \in I} V_i = V$. Given a non-zero vector $a \in V$ let

$$I(a) = \{\alpha \mid a \in V_\alpha\}.$$

Let us say that a representation has the *Gröbner property* if for each non-zero vector $a \in V$ the subset $I(a) \subseteq I$ has a minimum element $v(a)$. The *Gröbner map* is the map $v : V \setminus \{0\} \to I$ which associates to each non-zero vector the point $v(a) \in I$. The function v determines the representation, namely

$$V_\alpha = \{a \in V \mid v(a) \leq \alpha\}.$$

Let us say that a representation of an ordered set I with Gröbner property has *one-dimensional leaves*, if whenever $v(a) = v(b) = \alpha$, for $a, b \in V$ then there exist scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $v(\lambda_1 a + \lambda_1 b) > \alpha$.

Definition 4.1. A *pre-valuation* on a vector space V with values in the ordered set I is a representation of I, in the category of vector subspaces of V, with Gröbner property and one-dimensional leaves.

Example 4.2 (Functions with finite support on an ordered set)**.** Let $(I, >)$ be an ordered set. Consider the vector space V of complex-valued functions on I with finite support. For each $\alpha \in I$ let V_α be the subspace of V consisting of functions whose support is contained in the subset $I_\alpha = \{\beta \in I \mid \beta \leq \alpha\}$. The Gröbner map v associates to each non-zero function the smallest point in its support.

For a well-ordered ordered set I, i.e. a total order such that any non-empty subset of I has a minimum element, the condition of finiteness of support in Example 4.2 can be dropped.

Example 4.3 (Functions on a well-ordered ordered set)**.** Let I be a well-ordered set. Consider the vector space W of complex valued functions on I. For each element $\alpha \in I$ denote by W_α the subspace of functions whose support is contained in $I_\alpha = \{\beta \mid \beta \leq \alpha\}$. The Gröbner map associate to each non-zero function the smallest point in its support.

Let $L \in V$ be a finite dimensional subspace in a vector space V equipped with a pre-valuation with values in I.

Proposition 4.4. *The dimension of L is equal to the number of points in the image $v(L \setminus \{0\})$ under the Gröbner map $v : L \setminus \{0\} \to I$.*

Proof. We prove the claim by induction on the number k of elements in $v(L)$. Let $a \in L$ be such that $v(a)$ is minimum in $v(L)$. Using the condition about one-dimensional leaves of the representation one proves that the pre-image of $v(L) \setminus \{v(a)\}$ has codimension 1 in L. By induction hypothesis, the dimension of this pre-image is equal to $k-1$. The proposition is proved. $\square$

Example 4.5 (Schubert cells in Grassmannian)**.** Let $\mathrm{Gr}(n,k)$ be the Grassmannian of k-dimensional planes in $\mathbb{C}^n$. Let $I = \{1 < 2 \cdots < n\}$. In this case the space V from Example 4.2 naturally identifies with $\mathbb{C}^n$. Under the Gröbner map each k-dimensional subspace $V \subset \mathbb{C}^n$ goes to a subset $M \subset I$ containing k elements. The set of all k-dimensional subspaces which are mapped onto M form the *Schubert cell* X_M in the Grassmannian $\mathrm{Gr}(n,k)$. $\mathrm{GL}(n,\mathbb{C})$ naturally acts on $Gr(n,k)$ and the Schubert cells are in fact the orbits of the subgroup of upper triangular matrices.

Example 4.6 (Schubert cells in complete flag variety)**.** Let $F\ell_n$ be the variety of all complete flags $F = (L_0 \subset L_1 \subset \cdots \subset L_n = V)$ where $\dim(L_i) = i$. Similar to the Grassmannian, one defines cells for $F\ell_n$. The Schubert cells for $F\ell_n$ are parameterized by permutations $\sigma \in S_n$. Take a flag of subspaces $F = (L_0 \subset L_1 \subset \cdots \subset L_n = V)$. Let $M_i = v(L_i)$. By Proposition 4.4, $\#M_i = i$ and thus $\emptyset \subsetneqq M_0 \subsetneqq \cdots \subsetneqq M_n$. Let $\{\sigma(i)\} = M_i \setminus M_{i-1}$. Then $\sigma(F) = (\sigma(1), \ldots, \sigma(n))$ defines a permutation in S_n. Given a permutation $\sigma \in S_n$, $\{F \in F\ell_n \mid \sigma(F) = \sigma\}$ is the Schubert cell X_σ. As in the case of Grassmannian, the Schubert cells are the orbits of the group of upper triangular matrices with respect to the natural action of $\mathrm{GL}(n,\mathbb{C})$ on the flag variety.

Definition 4.7. Suppose the vector space V has the structure of a commutative algebra over $\mathbb{C}$ without zero divisors and the ordered set I has the structure of a commutative semigroup in which the addition and the ordering are compatible in a following sense: if $\alpha, \beta \in I$ with $\alpha > \beta$ then for any $\gamma \in I$, $\alpha + \gamma > \beta + \gamma$. A pre-valuation on the algebra V with values in I is called a *valuation* if for any two non-zero elements $a, b \in V$ we have

$$v(ab) = v(a) + v(b).$$

We will mostly be interested in the case when the algebra V is a field and the valuation take values in the ordered group $\mathbb{Z}^n$.

Example 4.8 (Ordering on the semigroup $\mathbb{Z}^n_+$ and the group $\mathbb{Z}^n$)**.** Let $\mathbb{Z}^n_+$ be the additive semigroup of integral points with non-negative coordinates in $\mathbb{Z}^n \subset \mathbb{R}^n$. One defines a well-ordering on $\mathbb{Z}^n$ as follows: fix $k \leq n$ independent linear functions $l_1, \ldots, l_k$ on R^n such that functions $\mathbf{f} : \mathbb{Z}^n_+ \to \mathbb{R}^k$, $\mathbf{l} = (l_1, \ldots, l_k)$ is one-to-one. This is always possible. Let us associate with each point $m \in \mathbb{Z}^n$ the ordered k-tuple of numbers $(l_1(m), \ldots, l_k(m))$

and define the ordering on $\mathbb{Z}^n$ using the lexicographic order in this set of k-tuples of numbers, namely, for $m_1, m_2 \in \mathbb{Z}^n_+$, we say that $m_1 > m_2$ if for some $0 \le i < k$, $l_1(m_1) = l_1(m_2), \dots, l_i(m_1) = l_i(m_2)$ and $l_{i+1}(m_1) > l_{i+1}(m_2)$. This gives a total ordering on $\mathbb{Z}^n$ compatible with addition, induced ordering on $\mathbb{Z}^n_+$ is a well-ordering. An ordering on $\mathbb{Z}^n$ compatible with addition is completely determined by the induced order on $\mathbb{Z}^n_+$, because for any two $m_1, m_2 \in \mathbb{Z}^n$ there is $m \in \mathbb{Z}^n$ such that $m_1 + m$ and $m_2 + m$ lie in $\mathbb{Z}^n_+$

A point $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_+$ can be identified with the monomial $x^m = (x_1^{m_1} \dots x_n^{m_n})$. Under this identification the addition in $\mathbb{Z}^n$ corresponds to the multiplication of monomials.

Example 4.9 (Gröbner ordering on the algebra of polynomials)**.** Under the identification of integral points and monomials, a function $c : \mathbb{Z}^n_+ \to \mathbb{C}$ with finite support corresponds to a polynomial $P(x_1, \dots, x_n) = \sum c(m) x^m$. Consider the well-ordered set $I = \mathbb{Z}^n$ with the ordering in Example 4.8. Using the pre-valuation in Example 4.2 on the set of functions with finite support we get a pre-valuation on the algebra of polynomials with a Gröbner mapping v from the set of non-zero polynomials to $\mathbb{Z}^n_+$. One verifies that this is compatible with multiplication of polynomials and is in fact a valuation.

This example can be naturally generalized to the algebra of power series.

Example 4.10 (Gröbner ordering on the algebra of formal power series and the algebra of germs of analytic functions)**.** To a function $c : \mathbb{Z}^n_+ \to \mathbb{C}$ one associates a formal power series $P(x_1, \dots, x_n) = \sum_{m \in \mathbb{Z}^n_+} c(m) x^m$. The construction Example 4.3 in this case gives a pre-valuation, together with a Gröbner ordering, on the algebra of formal power series. Again this is compatible with multiplication and is in fact a valuation.

Example 4.11 (Gröbner valuation on the field of rational functions)**.** Valuation in Example 4.9 can be extended to a valuation on the field of rational functions with values in the ordered group $\mathbb{Z}^n$. In fact each rational function R is a quotient of two polynomials $R = P/Q$. For $R \neq 0$ define $v(R) = v(Q) - v(P)$. This is well-defined, i.e. is independent of the choice of P and Q, and gives a valuation on the field the rational functions.

In the same way as in the previous example one defines a valuation on the quotient field of algebra of formal power series and on the field of germs of meromorphic functions.

We will use a valuation on the field of rational functions on an irreducible (quasi) affine algebraic variety X which take values in the group $\mathbb{Z}^n$. We say that a valuation is *faithful* if it is onto, i.e. takes all the values in $\mathbb{Z}^n$.

Example 4.12 (Gröbner valuation on the field of rational functions on an affine variety)**.** Let X be an irreducible n-dimensional (quasi) affine algebraic variety and let $f_1, \dots, f_n$ be regular functions on X. Assume that a some smooth point $a \in X$ is a common zero of all the f_i and their differentials

df_i at a are independent. Then in a neighborhood of the point a the functions $f_1, \dots, f_n$ define a local coordinate system on X. Fix a well-ordering ordering in the semigroup of monomials in the f_i which is isomorphic to the semigroup $\mathbb{Z}_+^n$. As in Examples 6,7, this Gröbner valuation can be be extended to a Gröbner valuation on the algebra of germs of analytic functions about the point a and to a Gröbner valuation on the field of meromorphic functions. In particular we obtain a faithful valuation on the field of rational functions on X with values in $\mathbb{Z}^n$.

One can modify valuations from the Example 4.12 to associate a similar valuation to a singular point on X.

Example 4.13 (Gröbner valuation on field of rational functions constructed from a Parshin point on X)**.** Consider a sequence of maps

$$\{a\} = X_0 \stackrel{\pi_0}{\to} X_1 \stackrel{\pi_1}{\to} \cdots \stackrel{\pi_{n-1}}{\to} X_n = X,$$

where each X_i, $i = 0, \dots n-1$, is a normal irreducible variety of dimension i and the map $X_i \stackrel{\pi_i}{\to} X_{i+1}$ is a normalization map for the image $\pi_i(X_i) \subset X_{i+1}$. Such a sequence represents a *Parshin point* on the variety X. A collection of rational functions $f_1, \dots, f_n$ represents *a system of parameters* about such the X_i, if for each i, the function $\pi_i^* \circ \cdots \circ \pi_n^* f_k$ on the hypersurface $\pi_{i-1}(X_{i-1})$ in the normal variety X_i has a zero of first order. Given a sequence of the X_i and a system of parameters, one can associate a *iterated Laurent series* to any rational function F. Iterated Laurent series can be defined inductively. It is a usual Laurent series $\sum_k c_k f_n^k$ with a finite number of terms with negative degrees in the variable f_n and each coefficient c_k in which is an iterated Laurent series in the variables $f_1, \dots, f_{n-1}$. Each iterated Laurent series has a monomial $f_1^{k_1} \dots f_n^{k_n}$ of the smallest degree with respect to the lexicographic order in degrees $(k_1, \dots, k_n)$ (where first we compare the degrees k_n, then the degrees k_{n-1} and so on). The map which assigns to a Laurent series its smallest monomial defines a faithful valuation on the field on rational functions on X.

4.2. **Hilbert Theorem.** Let X be an n-dimensional irreducible (quasi) affine algebraic variety and let $L \in K(X)$ be a finite dimensional vector space of regular functions on X. As in Section 2.1, the subspace L gives rise to a map

$$\Phi_L : X \to \mathbb{P}(L^*),$$

where L^* is the vector space dual of L. Let $Y = \Phi_L(X)$. The following is a version of the classical theorem of Hilbert. It plays a key role for us.

Theorem 4.14 (Hilbert's theorem)**.** *Let H be the Hilbert function of (X, L), defined by $H(k) = \dim L^k$. Then for large values of k, the function H becomes a polynomial in k. Moreover, the degree m of this Hilbert polynomial is equal to the dimension of the variety Y, and the leading coefficient c in the Hilbert polynomial is the degree of $Y \subset \mathbb{P}(L^*)$ divided by $m!$.*

Corollary 4.15. *For dimension $H(k)$ of the space L^k, there are numbers $0 \le m \le n$ and $c > 0$ such that*

$$\lim_{k\to\infty} \frac{H(k)}{k^m} = c.$$

2) m and c have the following properties: consider a system of equations $f_1 = \cdots = f_n = 0$ on the variety X, where $f_1, \dots, f_n$ are a general n-tuple of functions in L. If $m < n$ then the system has no roots on X. If $m = n$ then the system has $n!cd$ roots on X, where d is a mapping degree for $\Phi_L : X \to Y \subset \mathbb{P}(L^*)$.

4.3. **The graded semigroup and the Newton convex body of a subspace of regular functions: Main theorem.** Fix a faithful valuation v on the field of rational functions on a (quasi) affine irreducible variety X with values in the group $\mathbb{Z}^n$ (see Examples 9-10 in section 12). Using this valuation we associate a graded semi-group to each finite dimensional space of regular functions $L \subset K(X)$.

Definition 4.16. The *Gröbner semi-group* $G(L)$ of the space $L \in K(X)$ is the following semi-group.

$$G(L) = \bigcup_p \{(p, m) \mid m \in v(L^p \setminus \{0\})\} \subset \mathbb{Z} \times \mathbb{Z}^n.$$

Proposition 4.17. *1) For each space $L \in K(X)$ the Gröbner semi-group $G(L)$ is a graded semigroup in $\mathbb{Z} \times \mathbb{Z}^n$. The semigroup $G(L)$ has finite sections and a limited growth (see Section 3.3). 2) For any two spaces $L_1, L_2 \in K(X)$, the semigroup $G(L_1L_2)$ contains the semigroup $G(L_1) \oplus_t G(L_2)$ (see Section 3.4).*

Proof. 1) The number of points in the set $G(L)$ with the first coordinate equal to p is the dimension of the space L^p (Proposition 4.4). So the number of points in any section of $G(L)$ is finite and not equal to zero for every $p \in \mathbb{N}$. By definition points (p_1, m_1), (p_2, m_2) belong to $G(L)$, if for some functions $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$ we have $v(f_1) = m_1$, $v(f_2) = m_2$. The function $f_1 f_2$ belongs to the space $L^{p_1}L^{p_2}$ and $v(f_1 f_2) = m_1 + m_2$. So the point $(p_1 + p_2, m_1 + m_2)$ belongs to the set $G(L)$. Thus $G(L)$ is a graded semigroup with finite sections. According to Corollary 4.15 the semigroup $G(L)$ has limited growth.

2) By definition for each point (p, m) in the semigroup $G(L_1) \oplus_t G(L_2)$ there are points (p, m_1) and (p, m_2) in the semigroups $G(L_1)$ and $G(L_2)$ such that $m = m_1 + m_2$. And by definition of the semigroups $G(L_1)$ and $G(L_2)$ there are function $f_1 \in L_1^p$ and $f_2 \in L_2^p$ such that $v(f_1) = m_1$ i $v(f_2) = m_2$. The function $f_1 f_2$ belongs to $(L_1L_2)^p$ and $v(f_1 f_2) = m_1 + m_2$. Hence $G(L_1) \oplus_t G(L_2)$ is contained in $G(L_1L_2)$. □

Lemma 4.18. *Given a faithful valuation v on the field of rational functions on X with values in $\mathbb{Z}^n$ and a finite set $P \subset \mathbb{Z}^n$ with k elements, there exists*

a k-dimensional space $L \in K(X)$ of regular functions such that its image under v equals to $P + m$, for some $m \in \mathbb{Z}^n$.

Proof. Since the valuation is faithful, there is a finite dimensional space $\bar{L}$ of rational function such that its image under $v(\bar{L} \setminus \{0\}) = P$. We know the dimension of $\bar{L}$ equals k, the number of points in P. On can find a rational function g such that, after multiplication by g, all the functions in $\bar{L}$ by become regular functions. The image of the space $L = g\bar{L}$ is equal to $P + m$ where $m = v(g)$. □

Proposition 4.19. *For each space $L \in K(X)$ the semigroup $G(L)$ is contained in some graded semigroup of complete rank and with limited growth.*

Proof. Clearly if we enlarge a subspace L then its semi-group $G(L)$ becomes bigger (or remains the same). From Proposition 4.18 it follows that we can enlarge the space L (inside $K(X)$) so that the semigroup $G(L)$ becomes of complete rank. According to Proposition 4.17 all semigroups $G(L)$ for $L \in K(X)$ have limited growth. □

Let us summarize. Let X be an irreducible (quasi) affine variety of dimension n. Fix a faithful valuation v on the field of rational functions $\mathbb{C}(X)$ with values in $\mathbb{Z}^n$. The valuation v associate to a space $L \in K(X)$ the graded semigroup $G(L)$, which is contained in a graded semigroup of complete rank, and limited growth. To each such semigroup corresponds its Newton convex body $\Delta(G(L))$ and the index $ind(G(L))$ (Section 3.3). We are now ready to state our main theorem.

Theorem 4.20 (Main theorem)**.** *If the Newton convex body $\Delta(G(L)$ of the space $L \in K(X)$ has dimension n, then the intersection index $[L, \dots, L]$ of n copies of the space L is equal to*

$$n!V_n(\Delta(G(L))p(L)/ind(G(L)),$$

where $p(L)$ is the mapping degree $\Phi_L : X \to \mathbb{P}(L^*)$. If the Newton convex body $\Delta(G(L))$ has dimension smaller than n, then the image of X under the map Φ_L has dimension smaller than n and we have

$$[L, \dots, L] = 0.$$

The Newton convex body $\Delta(G(L_1L_2))$ of the product of two spaces $L_1, L_2 \in K(X)$ is contained in the Minkowski sum $\Delta(G(L_1)) + \Delta(G(L_2))$ of the Newton domains of those spaces. If the Newton domains $\Delta(G(L_1))$, $\Delta(G(L_2))$ have dimensions n, then $ind(G(L_1L_2))$ is not greater then each of the indices $ind(G(L_1))$ and $ind(G(L_2))$.

Proof. The main theorem is already proved. We reduced it to the description of the graded semigroups and their sums and to the Hilbert theorem. □

4.4. **Convex body associated to a line bundle over a projective variety.** Now let Y be an irreducible projective variety of dimension n and Ł an ample line bundle on Y. The ring of sections of Ł is defined as

$$R = \bigoplus_{k=0}^{\infty} H^0(Y, Ł^{\otimes k}).$$

Take a valuation $v : R \to \mathbb{Z}^n$ on the ring of sections. In fact, the constructions from Section 4.1, in particular, Example 4.13, applies in the same way to give valuation on the ring of sections.

Assign a semi-group $G(Ł)$ and a convex set $\Delta(G(Ł))$ to Ł in the same way as in Section 4.3, replacing the subspace L with $H^0(Y, Ł)$ and L^k with $H^0(Y, Ł^k)$.

Definition 4.21. The number $c_1(Ł)^n$, where $c_1(Ł)$ is the first Chern class, is called *degree* of Ł. It is equal to the number of solutions of a generic system $\sigma_1(y) = \cdots = \sigma_n(y) = 0$ of holomorphic sections of Ł.

As in Section 4.3, define the map

$$\Phi_Ł : Y \to \mathbb{P}(H^0(X, Ł)).$$

Since Ł is ample, the subvariety $\Phi_Ł(Y)$ has dimension n. If Ł is very ample then $\Phi_Ł$ is an embedding. Let $p(Ł)$ be the degree of the map $\Phi_Ł$. Let $ind(G(Ł))$ be the index of $G(Ł)$ in $\mathbb{Z} \times \mathbb{Z}^n$ (which is finite by the above assumption). The following is the analogue of Theorem 4.20 for projective varieties.

Theorem 4.22. *Let Y be a projective variety of dimension n and Ł an ample line bundle on Y. Let v be a valuation on R, the ring of holomorphic sections of Ł. Then*

$$\deg(Ł) = n! V_n(\Delta(G(Ł)) p(Ł) / ind(G(Ł)).$$

The proof is the same as the proof of Theorem 4.20, but instead of Hilbert theorem we use its projective version, namely asymptotic Riemann-Roch theorem.

Example 4.23 (Gelfand-Cetlin and string polytopes)**.** In representation theory, to any irreducible representation V_λ of $\mathrm{GL}(n, \mathbb{C})$, with highest weight λ, there corresponds a so-called *Gelfand-Cetlin polytope* Δ_λ. The integral points in this polytope parameterize the elements of a natural basis for V_λ. Using the heavy algebraic machinery of crystal graphs and canonical bases these construction has been generalized to any complex connected reductive algebraic group G. The resulting polytopes are called *string polytopes* (see for example [Litt]). Similarly the integral points in a string polytope, associated to a dominant weight λ, parameterize the elements of a natural basis for V_λ. It is shown in [Kaveh] that a string polytope corresponding to λ, in fact, coincides with the Newton polytope $\Delta(G(Ł_\lambda))$ for the flag variety $Y = G/B$ and the G-line bundle $Ł_\lambda$. The valuation v is a valuation coming

from a Parshin point corresponding to a sequence of Schubert varieties. The special case of $G = \mathrm{SP}(2n, \mathbb{C})$ has been proved by Okounkov earlier [Ok2].

One can identify the space of holomorphic sections $H^0(Y, \text{Ł})$ with a subspace of $\mathbb{C}(Y)$, and hence with a subspace of regular functions on an open (quasi) affine subvariety X of Y: Take s_0 to be a holomorphic section of the line bundle Ł with $D = \mathrm{Div}(s_0)$. Then any holomorphic section s is equal to fs_0 with $f \in \mathbb{C}(Y)$ satisfying $(f) + D \geq 0$. Thus $H^0(Y, \text{Ł})$ identifies with the subspace

$$L = L(D) = \{f \in \mathbb{C}(Y) \mid (f) + D \geq 0\}.$$

Similarly, for any k, $H^0(Y, \text{Ł}^k)$ identifies with $L(kD)$. Let X be an open (quasi) affine subvariety of $Y \setminus D$. Then all the functions in the $L(kD)$ are regular on X. Thus they can be regarded as subspaces of $\mathcal{O}(X)$, in fact, $L(kD) \in K(X)$. In general L^k is only a subset of $L(kD)$ and $L(kD)$ could be bigger than L^k. Thus the case of sections of a line bundle on a projective variety Y, while similar, is slightly different than the case of subspaces of regular functions on an (quasi) affine variety X.

Take a Parshin point

$$\{a\} = X_0 \overset{\pi_0}{\to} X_1 \overset{\pi_1}{\to} \cdots \overset{\pi_{n-1}}{\to} X_n = X,$$

in X. As in Example 4.13 this Parshin point gives a valuation on $\mathbb{C}(X)$ and the ring of sections R of any line bundle Ł. Since the divisor of s_0 is supported outside X then $v(s_0) = 0$ and hence $v(s) = v(fs_0) = v(f)$. That is, under the identification $s \mapsto f$, of sections of Ł with regular functions on X, the valuations on R and on $\mathbb{C}(X)$ agree.

Conversely, a (quasi) affine variety X and subspace $L \in K(X)$ gives rise to a projective variety Y and a line bundle Ł as follows: consider $\Phi_L : X \to \mathbb{P}(L^*)$. Put $Y = \overline{\Phi_L(X)}$ and let Ł be the line bundle on Y induced by the canonical line bundle $\mathcal{O}_{\mathbb{P}(L^*)}(1)$. As before fix a full valuation v on $\mathbb{C}(X) = \mathbb{C}(Y)$. Let L be very ample. The following theorem shows that the Newton convex body for (X, L) is the same as that of $(Y, \text{Ł})$.

Theorem 4.24. *$\Delta(G(\text{Ł})) = \Delta(G(L))$ and $ind(G(\text{Ł})) = ind(G(L))$.*

Proof. For every k, L^k can be identified with a subspace of $H^0(Y, \text{Ł}^{\otimes k})$. Thus $G(L) \subseteq G(\text{Ł})$ implying that $ind(G(L)) \geq ind(G(\text{Ł}))$ and $\Delta(G(L)) \subseteq \Delta(G(\text{Ł}))$. That is,

$$\frac{1}{ind(G(L))} V_n(\Delta(G(L)) \leq \frac{1}{ind(G(\text{Ł}))} V_n(\Delta(G(\text{Ł})).$$

But from definition we know $\deg(L) = \deg(\text{Ł})$ which by Theorems 4.20 and 4.22 implies that

$$n! V_n(\Delta(G(L))/ind(G(L)) = n! V_n(\Delta(G(\text{Ł}))/ind(G(\text{Ł}))$$

This shows that $ind(G(L)) = ind(G(\text{Ł}))$ and $\Delta(G(L)) = \Delta(G(\text{Ł}))$. $\square$

4.5. **Case of a Hamiltonian group action and relation with the moment polytope.** Let T be the algebraic torus $(\mathbb{C}^*)^k$ and Y a (smooth) projective T-variety equipped with a T-equivariant (very ample) line bundle Ł. The variety Y gets a symplectic structure from the projective embedding associated with Ł. Equipped with this symplectic structure, Y becomes a Hamiltonian $T_{\mathbb{R}}$-space where $T_{\mathbb{R}}$ is the real torus $(S^1)^k$. Let

$$\mu : Y \to \mathrm{Lie}(T_{\mathbb{R}})^*$$

be the moment map and $\mu(Y, Ł)$ be the moment polytope. Choose a T-stable Parshin point (see Section 4.13) namely a sequence

$$\{a\} = X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{n-1}} X_n = Y,$$

where each X_i, $i = 0, \dots n-1$, is a normal irreducible T-variety of dimension i and the map $X_i \xrightarrow{\pi_i} X_{i+1}$ is a normalization map for the image $\pi_i(X_i) \subset X_{i+1}$. Such a Parshin point always exists. Let v be the associated valuation. Let V and W denote the real span of $\Delta(G(Ł))$ and $\mu(Y, Ł)$ respectively. Put $\dim V = m$ and $\dim W = r$.

Theorem 4.25. [Ok1] *There is a natural linear map $\pi : V \to W$ with*

$$\pi(\Delta(G(Ł)) = \mu(Y, Ł).$$

Moreover, for every $\lambda \in \mu(Y, Ł)$, the $(m-r)$-dimensional volume of the fibre $\pi^{-1}(\lambda)$ is equal to the Duistermaat-Heckman piecewise polynomial measure $p(\lambda)$.

Remark 4.26. If the convex body $\Delta(G(Ł))$ is a polytope then the fact that Duistermaat-Heckman measure is equal to the $(m-r)$-dimensional volume of $\pi^{-1}(\lambda)$ implies its piecewise polynomiality.

In fact in [Ok1] the author considers the more general case of a reductive group G acting on Y. In this case he applies the construction of the convex set Δ to the smaller subalgebra of U-invariant functions (U being the maximal unipotent subgroup of G). As a result he obtains a convex set which is smaller than the convex set $\Delta(G(Ł))$ we considered. In particular, Okounkov's convex set in general could have dimension smaller than $\dim(Y)$.

5. Applications

5.1. **Theorems of Kushnirenko and Bernstein.** The well-known theorems of Kushnirenko and Bernstein are particular cases of Theorem 4.20. In this section we will discuss these theorems and show how they follow from Theorem 4.20. The proof below more or less the same as the proof of these results in [Khov1]. In fact the present paper should be considered as an unexpected and far-reaching generalization of [Khov1]). We will just sketch the proofs.

Let X be the affine variety $(\mathbb{C}^*)^n$ with the coordinates $x_1, \dots, x_n$. In the semigroup $K((\mathbb{C}^*)^n)$ of finite dimensional spaces of regular functions

on $(\mathbb{C}^*)^n$ there is a chosen subsemi-group $K_i((\mathbb{C}^*)^n)$, which contains all finite dimensional spaces, invariant under the group action. Theorems of Kushnirenko and Bernstein describe the intersection index in the semigroup $K((\mathbb{C}^*)^n)$ in geometrical terms. Let us start with definitions.

Each point $m = (m_1, \dots, m_n)$ in the group $\mathbb{Z}^n$ corresponds to the monomial $x^m = x_1^{m_1} \dots x_n^{m_n}$ (note that the monomials are the characters of the group $(\mathbb{C}^*)^n$). Each regular function f on the group $(\mathbb{C}^*)^n$ is a *Laurent polynomial*, i.e. is a linear combination of monomials $f = \sum c_m x^m$. To each regular function f one can associate its support $supp(f)$ — the finite set M in the group $(\mathbb{C}^*)^n$ which consists of all points m, such that the monomial x^m appears in the representation of the function f in the form of Laurent polynomial $f = \sum_{m \in M} c_m x^m$, with a non zero coefficient $c_m \neq 0$. The Newton polyhedron $\Delta(f) \subset \mathbb{R}^n$ of f is the convex hall of the support $supp(f) \subset \mathbb{Z}^n \subset \mathbb{R}^n$ of the function f. For each finite set $M \subset \mathbb{Z}^n$ denote by $L(M)$ the vector space of Laurent polynomials f, which support belongs to the set M, $sup(f) \subset M$. Fix any Gröbner ordering on the lattice $\mathbb{Z}^n$. One can easily proof the following statement

Proposition 5.1. *Each space in the semigroup $K_i((C^*)^n)$ is a space $L(M)$ for some finite subset M in the lattice $\mathbb{Z}^n$. The Gröbner map maps the space $L(M)$ into the set M (in particular, the image of the space L under the Gröbner map is independent on a Gröbner ordering). For each couple of spaces $L_1, L_2 \in K_i((C^*)^n)$ the following relation holds $Gr(L_1L_2) = v(L_1) + v(L_2)$. The Newton convex body of the space $L \in K_i((C^*)^n)$ coincides with the convex hall of the set M.*

Assume that the convex hall of the set M has the dimension n. It is easy to see that the degree of the natural map $(C^*)^n \to PL(M)^*$ is equal to the index of the subgroup in $\mathbb{Z}^N$ generated by the set M.

Using these facts one can reduce from the main theorem the following results

Theorem 5.2 (Kushnirenko)**.** *Let $f_1, \dots, f_n$ be a generic n -tuple of Laurent polynomials with fixed Newton polyhedra Δ. Then the number of the roots on $(\mathbb{C}^*)^n$ of the system $f_1 = \cdots = f_n = 0$ is equal to $n!V(\Delta)$.*

Theorem 5.3 (Bernstein)**.** *Let $f_1, \dots, f_n$ be a generic Laurent polynomials with the Newton polyhedra $\Delta_1, \dots, \Delta_n$. Then the number of the roots on $(\mathbb{C}^*)^n$ of the system $f_1 = \cdots = f_n = 0$ is equal to multiplied by $n!$ Minkowski mixed volume of this polyhedra, e.i. is equal to $n!V(\Delta_1, \dots, \Delta_n)$.*

5.2. **Brunn–Minkowski inequality, its corollaries and generalizations.** Let $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ be bounded convex bodies and let $\Delta = \Delta_1 + \Delta_2$ be their Minkowski sum. The following important and simple inequality was discovered by Brunn.

Theorem 5.4 (Brunn–Minkowski inequality)**.**

$$V_n^{\frac{1}{n}}(\Delta_1) + V_n^{\frac{1}{n}}(\Delta_2) \leq V_n^{\frac{1}{n}}(\Delta).$$

When $n = 2$, i.e. on the plane, the Brunn–Minkowski inequality has the following form.

Theorem 5.5 (Isoperimetric inequality for planar regions)**.** *Areas* V_2 *of the bounded convex planar bodies* Δ_1 *and* Δ_2 *and their mixed area* $V_2(\Delta_1, \Delta_1)$ *are related by the following inequality* $V_2(\Delta_1)V_2(\Delta_2) \leq V_2^2(\Delta_1, \Delta_2)$.

Proof. From the Brunn–Minkowski inequality we have the following relation $(V_2^{\frac{1}{2}}(\Delta_1)+V_2^{\frac{1}{2}}(\Delta_2))^2 \leq V_2(\Delta_1+\Delta_2) = V_2(\Delta_1)+2V_2(\Delta_1, \Delta_2)+V_2(\Delta_2)$, which is equivalent to $V_2(\Delta_1)V_2(\Delta_2) \leq V_2^2(\Delta_1, \Delta_2)$. $\square$

Let $l(\partial\Delta)$ denote the perimeter of a two dimensional convex body Δ. Let B_1 be the unit ball centered at the origin. It is easy to see that $V_2(\Delta, B_1) = \frac{1}{2}l(\partial\Delta)$. If $\Delta_2 = B_1$ the isoperimetric inequality becomes

$$V_2(\Delta)V_2(B_1) \leq \frac{1}{4}l^2(\partial\Delta),$$

and hence

$$V_2(\Delta) \leq \frac{1}{2}l^2(\partial\Delta)/\pi.$$

Thus we obtain an estimate of the area $V_2(\Delta)$ of the body Δ) in terms of its perimeter. This classical estimate known as *isoperimetric inequality* is sharp. That is why the two dimensional case of the Brunn – Minkowski inequality is also called isoperimetric inequality.

The Brunn–Minkowski inequality has numerous generalizations. Some of them (e.g. Alexandrov–Fenchel inequality), have rather complicated proofs. Here we list some generalizations and corollaries of the Brunn-Minkowski inequality.

Theorem 5.6 (Alexandrov–Fenchel inequality)**.** *Let* $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$ *be bounded convex bodies in* $\mathbb{R}^n$*. Denote by* $V_n(\Delta)$ *the volume of* Δ *and by* $V_n(\Delta_1, \dots, \Delta_n)$ *the mixed volume of* $\Delta_1, \dots, \Delta_n$*. The following inequality holds*

$$V_n(\Delta_1, \Delta_2, \Delta_3 \dots, \Delta_n)^2 \geq V_n(\Delta_1, \Delta_1, \Delta_3 \dots, \Delta_n)V_n(\Delta_2, \Delta_2, \Delta_3 \dots, \Delta_n).$$

The following inequalities are formal corollaries from the Alexandrov–Fenchel inequality.

Corollary 5.7 (Corollaries of Alexandrov–Fenchel inequality)**.** *Let* P*,* Q *and* Δ_i *be bounded convex bodies in* $\mathbb{R}^n$*. The following inequalities hold:*
(a) $V(\Delta_1, \dots, \Delta_n)^m \geq \prod_{i=1}^m V(\underbrace{\Delta_i, \dots, \Delta_i}_{m}, \Delta_{m+1}, \dots, \Delta_n)$.

(b) $V(\Delta_1, \dots, \Delta_n) \geq \mathrm{Vol}(\Delta_1) \cdots \mathrm{Vol}(\Delta_n)$.

(c) $V(\underbrace{P, \dots, P}_{i}, \underbrace{Q, \dots, Q}_{m-i}, \Delta_{m+1}, \dots, \Delta_n) \geq V(\underbrace{P \dots, P}_{m}, \Delta_{m+1}, \dots, \Delta_n)^i \cdot V(\underbrace{Q, \dots, Q}_{m}, \Delta_{m+1}, \dots, \Delta_n)^{m-i}$.

(d) $V(\underbrace{P,\dots,P}_{k},\underbrace{Q,\dots,Q}_{l},\Delta_{k+l+1},\dots,\Delta_n)^2 \geq$
$V(\underbrace{P,\dots,P}_{k-1},\underbrace{Q,\dots,Q}_{l+1},\Delta_{k+l+1},\dots,\Delta_n)\cdot V(\underbrace{P,\dots,P}_{k+1},\underbrace{Q,\dots,Q}_{l-1},\Delta_{k+l+1},\dots,\Delta_n).$

5.3. **Algebraic analogue of Brunn–Minkowski and Alexandrov–Fenchel inequalities and their corollaries.** Let $L_1, L_2 \in K(X)$ be finite dimensional spaces of regular functions on an irreducible (quasi) affine variety X. Consider the self-intersection indices $[L_1,\dots,L_1]$, $[L_2,\dots,L_2]$ and $[L_1L_2,\dots,L_1L_2]$ of the spaces L_1, L_2 and L_1L_2. Assume that the spaces L_1 and L_2 separate generic points on X, i.e. the mapping degrees of the maps Φ_{L_1} and Φ_{L_2} are 1.

Theorem 5.8 (Analogue of Brunn–Minkowski inequality for self-intersection index)**.** *If the spaces L_1 and L_2 separate generic point on an irreducible (quasi) affine algebraic variety X, then we have*

$$[L_1,\dots,L_1]^{\frac{1}{n}} + [L_2,\dots,L_2]^{\frac{1}{n}} \leq [L_1L_2,\dots,L_1L_2]^{\frac{1}{n}}.$$

Proof. The inequality follows from Theorem 4.20 and the classical Brunn–Minkowski inequality: L_1 and L_2 separate generic points on X and hence dimensions of their Newton convex bodies $\Delta(G(L_1))$ and $\Delta(G(L_2))$ are equal to n. Newton convex body $\Delta(G(L_1L_2))$ of the space L_1L_2 contains $\Delta(G(L_1))+\Delta(G(L_2))$. Also index of the group generated by the semigroup $G(L_1L_2)$ is not bigger than minimum of indices of the groups generated by the semigroups $G(L_1)$ and $G(L_2)$. Applying the Brunn–Minkowski inequality to the Newton convex bodies $\Delta(G(L_1))$, $\Delta(G(L_2))$ and $\Delta(G(L_1L_2))$ we get the required inequality. □

When X is a surface (i.e. $n=2$) from the algebraic analogue of Brunn–Minkowski one obtain the following.

Theorem 5.9 (Affine version of Hodge Index Theorem)**.**

$$[L_1,L_1][L_2,L_2] \leq [L_1,L_2]^2.$$

Proof. From Theorem 5.8 we have $([L_1,L_1]^{\frac{1}{2}}+[L_2,L_2]^{\frac{1}{2}})^2 \leq [L_1L_2,L_1L_2] = [L_1,L_1]+2[L_1,L_2]+[L_2,L_2]$, and this is equivalent to the inequality $[L_1,L_1][L_2,L_2] \leq [L_1,L_2]^2$. □

In Section 2.7 using the affine version of Hodge Index Theorem we proved the following.

Theorem 5.10 (Algebraic analogue of Alexandrov–Fenchel inequality)**.** *Let X be an irreducible n-dimensional (quasi) affine variety and $L_1,\dots,L_n \in K(X)$ very ample spaces. Then*

$$[L_1,L_2,L_3\dots,L_n]^2 \geq [L_1,L_1,L_3\dots,L_n][L_2,L_2,L_3\dots,L_n].$$

The following are formal corollaries from the algebraic analogue of Alexandrov–Fenchel inequality.

Corollary 5.11 (Corollaries of the algebraic analogue of Alexandrov–Fenchel inequality)**.** *Let X be an n-dimensional (quasi) affine irreducible variety and let P, Q and L_i be very ample spaces from the semigroup $K(X)$. The following hold:*
(a) $[L_1, \dots, L_n]^m \geq \prod_{i=1}^m [\underbrace{L_i, \dots, L_i}_{m}, L_{m+1} \dots, L_n]$;
(b) $[L_1, \dots, L_n] \geq [L_1, \dots, L_1] \dots [L_n, \dots, L_n]$;
(c) $[\underbrace{P, \dots, P}_{i}, \underbrace{Q, \dots, Q}_{m-i}, L_{m+1}, \dots, L_n] \geq [\underbrace{P, \dots, P}_{m}, L_{m+1}, \dots, L_n]^i \cdot$
$[\underbrace{Q, \dots, Q}_{m}, L_{m+1}, \dots, L_n]^{m-i}$;
(d) $[\underbrace{P, \dots, P}_{k}, \underbrace{Q, \dots, Q}_{l+1}, L_{k+l+1}, \dots, L_n]^2 \geq$
$[\underbrace{P, \dots, P}_{k-1}, \underbrace{Q, \dots, Q}_{l+1}, L_{k+l+1}, \dots, L_n][\underbrace{P, \dots, P}_{k+1}, \underbrace{Q, \dots, Q}_{l-1}, L_{k+l+1}, \dots, L_n]$.

Inequalities (a)-(d) follow from the algebraic analogue of the Alexandrov–Fenchel inequality exactly in the same way as the similar geometrical inequalities follows from the Alexandrov–Fenchel inequality.

Remark 5.12. Of course all the previous inequalities hold for the case of a projective variety and an ample line bundle (with identical proofs).

Using the classical Brunn-Minkowski inequality it is elementary to see that the function $\Delta \mapsto \log(\mathrm{Vol}(\Delta))$ is a concave function on the space of convex bodies in $\mathbb{R}^n$. Let Y be a projective variety with a line bundle Ł. From analogue of Theorem 5.8 for $(Y, Ł)$ it follows that

Corollary 5.13 (Log-concavity of degree of line bundles)**.** *The function*

$$Ł \mapsto \log(\deg(Ł)),$$

is a concave function.

5.4. **Algebraic proof of the geometric inequalities.**

Theorem 5.14. *All geometric inequalities from Section 5.2 follow from their algebraic analogues in Section 5.4.*

Proof. It it enough to prove the Alexandrov–Fenchel inequality. For polyhedra with integral vertices it follows from its algebraic analogue and from Bernstein theorem which states that the mixed volume of such polyhedra is equal to the intersection number of spaces L_i in $K(X)$ where $X = (\mathbb{C}^*)^n$ and L_i are spaces of Laurent polynomials with fixed Newton polyhedra (Section 5.1). The inequality for the convex polyhedra with rational vertices follows from this result because, after multiplication by an appropriate number, one turn a polyhedron with rational vertexes into a polyhedron with integral vertices. Now it is enough to use the inequality for polyhedra with integral vertexes and multi-linearity of the mixed volume. Finally any convex body

can be approximated, in the Hausdorff metric, by polyhedra with rational vertices. Since mixed volume is continuous with respect to the topology induced by the Hausdorff metric the theorem is proved. □

Note that an algebraic proof of the Alexandrov–Fenchel inequality has been known earlier (see [Burago-Zalgaller, Addendum 3, Algebra and mixed volumes, pp. 182-207]). But a crucial step in that proof is the use of Hodge Index Theorem. In fact the proof of Hodge Index Theorem itself is as complicated as the Alexandrov–Fenchel inequality. Here we have given rather simple proofs of both of this two theorems (Hodge Index Theorem and Alexandrov–Fenchel inequality) simultaneously using just two very classical results: Brunn–Minkowski inequality (and in fact it is even enough to use the isoperimetric inequality in the plane) in the geometric side and the Hilbert theorem on degree in the algebraic side.

5.5. **Newton convex body in the case of curves.** In this section we assume that X is an affine irreducible curve. As before we say that the space $L \in K(X)$ is *ample* if the mapping degree for the natural map $\Phi_L : X \to \mathbb{P}(L^*)$ is 1, in other words if Φ_L, restricted to an open dense subset, is an embedding. Using Hilbert theorem and Riemann–Roch theorem one can find a good estimate for the dimension of the space L^k for large values of k.

Proposition 5.15. *Assume that space $L \in K(X)$ is ample. Then there exists $k_0 \geq 0$ and $C \geq 0$ such that for $k > k_0$ the dimension of the space L^k is equal to $k \deg L + C$. Moreover one can take $C \leq 1 - g$ where g is the genus of the curve X.*

Proof. The dimension of the curve X is 1 and the number of zero of a generic function $f \in L^k$ is equal to $\deg L^k = k \deg L$ (see the proposition 5.1). So by Hilbert theorem, for sufficiently big k, the dimension of the space L^k is equal to $k \deg L + C$. Consider a compactification $\bar{X}$ of X. At any point $a \in A = \bar{X} \setminus X$ we have $ord_a L^k = k ord_a L$. By Riemann–Roch theorem if $ord_a L^k > 2g - 2$ the dimension of the space of all regular functions on X whose order at the point $a \in A$ is $\geq ord_a L^k$ is equal to $\sum_{a \in A} ord_a L^k - g = k \dim L - g + 1$. □

Definition 5.16. Let $a \in X$. The *valuation corresponding to a* is the valuation v_a on $\mathbb{C}(X)$ defined by $v_a(f) = ord_a f$ for any $f \in \mathbb{C}(X)$. This valuation take values in $\mathbb{Z}$ is faithful: for each integer $m \in \mathbb{Z}$ there is a rational function f with $ord_a f = m$.

For an ample space L and for the valuation v_a corresponding to a point $a \in X$ one can find a good description of the graded semigroup $G(L)$ and the Newton convex body $\Delta(G(L))$. For a $k > 0$ denote by $G_k(L)$ the set of degree k elements in the semigroup $G(L)$. Denote by $\tilde{G}(L)$ the projection of the $G(L)$ on the valuation line.

Theorem 5.17. *If a space L is ample, then for a valuation related to a point $a \in X$ the following holds.*

(1) *The semigroup* $\tilde{G}(L)$ *generates the group* $\mathbb{Z}$.
(2) *The projection of the Newton convex body* $\Delta(G(L))$ *of the graded semigroup* $G(L)$ *on the valuation line is the segment* $[0, \deg L]$. *For any* $k > 0$, *each point* (k, m) *in the set* $G_k(L)$ *satisfies the inequalities* $0 \leq m \leq k \deg L$.
(3) *There is a constant* C_0 *and a function* $C_1(k)$ *such that* $\lim_{k\to\infty} \frac{C_1(k)}{k} = 0$ *and the set* $G_k(L)$ *contains all points* (k, m) *satisfying* $C_0 \leq m \leq \deg L - C_1$.

Proof. Since L is ample the map $\Phi_L : X \to \mathbb{P}(L^*)$ induces a birational isomorphism between the curve X and its image Y. This means that each rational function on X is pull-back of a rational function on Y. So $\tilde{G}(L) = \mathbb{Z}$. By Theorem 4.20 (main theorem) the number of roots on X of a generic function $f \in L$ is equal to the length of the Newton segment $\Delta(G(L))$ (note that a 1-dimensional bounded convex domain is a segment). On the other hand this number of roots is equal to $\deg L$ (see Proposition 2.16). So the length of the segment $\Delta(G(L))$ is equal to $\deg L$.

The space L contains a function f with $f(a) \neq 0$. So the semigroup $G(L)$ contains the point $e = (1, 0)$. Hence the projection of the Newton segment on the valuation line contains the origin and it coincides with the segment $[0, \deg L]$. As we proved above (see ...) there are two non-negative functions $C_0(k)$ $C_1(k)$ such that $\lim_{k\to\infty} \frac{C_0(k)}{k} = \lim_{k\to\infty} \frac{C_1(k)}{k} = 0$ and any point satisfying the inequalities $C_0(k) \leq m \leq \deg L - C_1(k)$ belongs to $G_k(L)$. For $k_2 > k_1$, by adding vector $(k_2 - k_1)$, we can embed $G_{k_1}(L)$ into $G_{k_2}(L)$. So the condition $C_0(k) \leq m$ can be replaced by $C_0 \leq m$, where C_0 is a sufficiently large positive constant. □

Fix a valuation corresponding to some point $a \in X$.

Corollary 5.18. *If the space* L *is ample, then the number* $[L]$ *of roots of a sufficiently general function* $f \in L$ *is equal to the length of the Newton segment* $\Delta(G(L))$. *The Newton segment* $\Delta(G(L_1L_2))$ *of the product of two ample enough spaces* L_1 *and* L_2 *is equal, up to a shift, to the sum* $\Delta(G(L_1)) + \Delta(G(L_2))$ *of the Newton segments of* L_1 *and* L_2.

The semigroup $G(L)$ belongs to a cone over a segment which contains points $(1, x)$ where $0 \leq x \leq \deg L$. Clearly the cone of the semi-group $G(L)$ consists of two rays one of which is the upper-half of the vertical axes. The following question is important for us: does the semigroup $G(L)$ contain an integral point on the other boundary ray of its cone? that is, does $G(L)$ contain an integral point at the ray $\lambda(1, \deg L)$, $\lambda > 0$? If the genus g of the curve X is positive, then as a rule the answer to the question is negative. Indeed we have the following. let D be a divisor of poles of the space L and a be the point corresponding to the valuation.

Proposition 5.19. *If the semigroup* $G(L)$ *contains a point* $(k, k \deg L)$ *where* $k > 0$ *then the divisors* kD *and* $(k \deg L)a$ *are linearly equivalent.*

Proof. If the point $(k, k\deg L)$ belongs to $G(L)$ then there is $f \in L^k$ such that the divisor of its poles is kD and the divisor of its zeros is $(k\deg L)a$. The existence of such a function means that $k(\deg L)a - kD$ is a principal divisor. □

Corollary 5.20. *If the genus g of X is positive and if the space $L \in K(X)$ is ample then for almost all points $a \in X$, the semigroup $G(L)$, for the valuation corresponding to a, is not finitely generated.*

Proof. If a semigroup is generated by a finite set $M = \{(k, m)\}$, where $k > 0$, then a point in M at which the quotient m/k attains its maximum belongs to the boundary of the minimal convex cone which contained the semigroup. By the proposition 19.4 it could happen only if for some natural k the divisor $(k\deg L)a$ is equivalent to the divisor kD. At most countable set of points $a \in X$ could satisfy this condition if the genus g is positive. □

Let L be an element in the semigroup $K(X)$ and let $Y = \Phi_L(X)$ be the image of X in $\mathbb{P}(L^*)$. Let d denote the mapping degree of $\Phi_L : X \to Y$ and let μ_a be the local mapping degree of Φ_L at a point a. Fix the valuation related to the point a.

Theorem 5.21. *With notation as above, the following are true:*

(1) *The group generated by the semigroup $\tilde{G}(L)$ is a subgroup of index μ_a in $\mathbb{Z}$.*
(2) *The projection of the Newton convex body $\Delta(G(L))$ of the graded semigroup $G(L)$ on the valuation line is the segment $[0, \frac{\mu_a \deg(L)}{d}]$. For any $k > 0$, every point (k, m) in the set $G_k(L)$ satisfies the inequality*
$$0 \leq m \leq \frac{k\mu_a \deg L}{\deg},$$
moreover the number m is divisible by μ_a.
(3) *There is a constant C_0 and there is a function $C_1(k)$ of such that*
$$\lim_{k\to\infty} \frac{C_1(k)}{k} = 0,$$
and the set $G_k(L)$ contains all points (k, m) such that m is divisible by μ_a and $C_0 \leq m \leq \deg L - C_1$.

Proof. The field of rational functions on Y is isomorphic to the field of rational functions on a non singular model $\tilde{Y}$ of the curve Y. The map Φ_L can be lifted to the map $\tilde{\Phi}_L : X \to \tilde{Y}$. The germ of the curve X at the point a covers the germ of the curve $\tilde{Y}$ at the point $\tilde{\Phi}_L(a)$ with the multiplicity μ_a. Because of this the index of the subgroup of $\mathbb{Z}$ generated by the semigroup $\tilde{G}(L)$ is equal to μ_a.

By Theorem 4.20 (main theorem) the number of zeros on X of a generic function $f \in L$ is equal to the length of the Newton segment $\Delta(G(L))$ multiplied by the number d/μ_a. On the other hand this number of roots is

equal to $\deg(L)$. So the length of the segment $\Delta(G(L))$ is equal to $\frac{\mu \deg(L)}{d}$. To finish the proof use the same arguments in the proof of Theorem 5.17. $\square$

Corollary 5.22. *For each space $L \in K(X)$ and for the valuation corresponding to a point $a \in X$ we have $d \geq \mu$, where d is the mapping degree of $\Phi_L : X \to \mathbb{P}(L^*)$ and μ is the index of the subgroup of $\mathbb{Z}$ generated by generated by the semigroup $\tilde{G}(L)$.*

Remark 5.23. 1) In general the numbers d and *ind* in the main theorem are different. But when X is a so-called spherical variety for the action of a connected reductive algebraic group G and the space $L \in K(X)$ is invariant under G these two numbers coincide (see [K-Kh]). 2) The inequality in Corollary 5.22 can be easily extended to the general higher dimensional case.

5.6. **Degeneration of a variety to a toric variety and SAGBI bases.** Let $A = \mathbb{C}[x_1, \ldots, x_n]$. Fix a term ordering $<$ on $\mathbb{Z}^n$ which we regard as the semi-group of monomials in A. As usual define the initial term map $v : R \to \mathbb{Z}^n$ as follows: let $f \in A$ and let $cx_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be the lowest term of f with respect to $<$. Put $v(f) = (\alpha_1, \ldots, \alpha_n)$. v extends to a valuation on the field of rational polynomials $\mathbb{C}(x_1, \ldots, x_n)$. Let R be a subring of A. R is said to have a SAGBI basis (subalgebra analogue of Gröbner basis for ideals), with respect to $<$, if the semi-group of initial terms $v(R)$ is finitely generated. A set of polynomials $\{f_1, \ldots, f_r\} \subset R$ such that $v(f_1), \ldots, v(f_r)$ is a set of generators for the semi-group $v(R)$ is called a SAGBI basis. The remarkable fact about a SAGBI basis is that any element of R can be represented as a polynomial in the f_i by a classical simple finite algorithm called *subdection algorithm.*

Below we generalize the notion of SAGBI basis to a finitely generated subalgebra of the coordinate ring of a (quasi) affine variety. Let X be a (quasi) affine variety of dimension d. Fix a term order on $\mathbb{Z}^n$ and let $v : \mathbb{C}(X) \to \mathbb{Z}^n$ be a valuation with respect to $<$. As usual let L be a finite dimensional subspace of $\mathcal{O}(X)$ and put $R = \bigoplus_{k=0}^{\infty} L^k$.

Definition 5.24. R is said to have a SAGBI basis, with respect to v, if $v(R)$ is a finitely generated semi-group. Similarly, let Y be a projective variety and Ł a line bundle. Put $\mathcal{R} = \bigoplus_{k=0}^{\infty} H^0(Y, Ł^k)$. $\mathcal{R}$ is said to have a SAGBI basis if $v(\mathcal{R})$ is a finitely generated semi-group.

Obviously if R (respectively $\mathcal{R}$) has a SAGBI basis then the valuation cone of v is a convex polyhedral cone and the convex body $\Delta(G(L))$ (respectively $\Delta(G(Ł)))$ is a polytope. Conversely, if the valuation cone of v is polyhedral and moreover if $v(R)$ (respectively $v(\mathcal{R})$) coincides with all the integral points in the valuation cone then it is a finitely generated semi-group and hence R (respectively $\mathcal{R}$) has a SAGBI basis. This is the case in many important examples namely, toric varieties, flag varieties and spherical varieties of a complex connected reductive group G(See [Kaveh]).

When the homogeneous coordinate ring $\mathcal{R}$ of a projective variety $Y \subset \mathbb{P}(V)$ has a SAGBI basis, it follows from [Eisenbud, Theorem 15.17] that Y can be degenerated to a toric variety, that is, there is a flat family $Y_t, t \in \mathbb{C}$ of subvarieties of $\mathbb{P}(V)$ such that Y_t is isomorphic to Y for $t \neq 0$ and Y_0 is a toric variety (with the same dimension as Y of course).

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